UNIT I - MATRICES

Introduction

Matrices have become an important tool in the study of science and engineering. Particularly it have been found great utility in the theory of electrical circuits, mechanics, cryptography etc. The evolution of computers have facilited the wide use of matrices in applied mathematics. We confine ourselves to the study of eigenvalue problems.

A set of mn numbers (real or imaginary) arranged in a rectangular array of m rows and n columns is called an $m\times n$ **Matrix**. It is usually written as $A=(a_{ij})_{m\times n}$. Then $m\times n$ is said to tbe **order** of the matrix A.

Example: $A=\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix}$ is a matrix of order 3×3. $A=\begin{pmatrix}1&2&0\\3&1&2\end{pmatrix}$ is a matrix of order 2×3.

The sum of the elements in the principal diagonal of a square matrix is called the **trace** of the matrix.

Example: If $A=\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix}$, then trace of the matrix is 1+3+2 = 6.

A square matrix A is **symmetric** if $A^T=A$ and **skew symmetric** if $A^T=-A$

Example for symmetric matrix:

$$A=\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix}$$

Example for skew-symmetric matrix:

$$\mathrm{A}=\begin{pmatrix}0&1&-2\\-1&0&3\\2&-3&0\end{pmatrix}$$

The matrix obtained from a given matrix A by interchanging rows and columns is called the transpose of A and is denoted by A^T.

Example: If $\mathrm{A}=\begin{pmatrix}0&1&-2\\-1&0&3\\2&-3&0\end{pmatrix}$, then $\mathrm{A}^{\mathrm{T}}=\begin{pmatrix}0&-1&2\\1&0&-3\\-2&3&0\end{pmatrix}$.

Some Properties of Matrices:

1. The non singular matrix A is **symmetric** then A^{-1} is also symmetric.
2. A square matrix of order n is said to be **orthogonal** if $AA^T=A^TA=I_n$. Then $\mathrm{A}^{\mathrm{T}}=\mathrm{A}^{-1}$.
3. If A is **orthogonal**, then A is non singular.
4. If A is an orthogonal matrix, then A^T, A^{-1} are also orthogonal matrix.
5. A square matrix A is said to be **singular** if $|A|=0$. and it is called **non singular** if $|A|\neq 0$

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Example: Let $A = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}$.

Then $|A| = \begin{vmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} = 0(0+9) - 1(0-6) - 2(3-0) = 6 - 6 = 0.$

Hence A is singular.

Example: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0.$

Hence A is non singular.

Suppose A is a non singular matrix of order n. If there exists a matrix B such that $AB = BA = I_n$

then B is called the **multiplicative inverse** of A and it is denoted by $B = A^{-1}$ where

$$A^{-1} = \frac{1}{|A|} Adj(A)$$

Rank of a Matrix

If A is a matrix of order $m \times n$, then the **rank** of A is said to be 'r' if (i) there exists at least one minor of order 'r' which does not vanish (ii) every minor of order (r+1) or higher orders vanish. Clearly $r(A) \leq (m,n)$. i.e. the rank of the matrix is the largest of the order of all non vanishing minors of A.

Another method to find the rank

Reduce A to any one of the forms $[I_r]$, $[I_r \mid 0]$, $\begin{bmatrix} I_r \\ \hline 0 \end{bmatrix}$, $\begin{bmatrix} I_r & o \\ 0 & 0 \end{bmatrix}$ by a series of elementary operations on A and then find the order of the unit matrix contained in the normal form of A. Elementary operations are (i) interchange of any two rows(or columns) (ii) multiplication of any row(or column) by a non zero scalar (iii) addition of any row(or column), the same scalar multiplies of any other row(or column).

Example: Find the rank of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1(6-1) - 2(2-9) + 3(2-9) = -2 \neq 0.$$

Therefore $r(A) = 3$

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Example: Find the rank of $A=\begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$.

$$|A|=\begin{vmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{vmatrix}=2(2-4)-3(4+2)+4(6+1)=0.\ \textit{Therefore}\ r(A)\neq 3$$

But there is at least one non zero minor $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}\neq 0$ *Therefore* $r(A)=2.$

The Characteristic Equation and Characteristic Root of a Matrix

Let A be a square matrix of order n and λ be any scalar. Then the equation $|A-\lambda I|=0$ is called characteristic equation of A with degree n. The roots of this equation are called characteristic roots or latent roots or **eigenvalues** of A.

Let A be a square matrix. If there exists a non zero column vector X and a scalar λ such that $AX=\lambda X$, then X is called the **eigenvector** corresponding to the eigen value λ.

Remark:

If all the eigenvalues are distinct then the corresponding eigenvectors are **linearly independent**.

If two or more eigenvalues are equal then the eigenvectors may be **linearly dependent or independent**.

For a given one eigenvalue, we can have more than one eigenvectors. i.e. kX is also a solution.

Properties of Eigenvalues

1. Sum of the eigenvalues of a square matrix A is the sum of diagonal elements of A.
2. Product of eigenvalues of a square matrix A is the value of determinant of A.
3. The eigenvalues of A and its transpose A^T are the same.
4. The eigenvalues of a triangular matrix or diagonal matrix are precisely the diagonal elements of the matrix.
5. The eigenvalue of kA are k times the eigenvalue of A, k being a scalar.
6. If λ is the eigenvalue of A, then λ^k is the eigenvalue of A^k, k is a positive integer.
7. The eigenvalues of a real symmetric matrix are real.
8. The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.
9. If A is an orthogonal matrix , with eigenvalue λ, then $\frac{1}{\lambda}$ is also an eigen value of A
10. To a eigenvector of a matrix, there cannot correspond two different eigenvalues, but to a eigenvalue there corresponds different eigenvectors.
11. Two matrices A and $P^{-1}AP$ have same eigenvalues.
12. If A is a non singular matrix with eigenvalue $\lambda(\neq 0)$. Then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and $\frac{|A|}{\lambda}$ is an eigenvalue of adj(A).

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Note 1: Let A be a 3×3 matrix. Then the expansion of characteristic equation $|A-\lambda I| = 0$

becomes $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of diagonal values of A

$S_2 =$ sum of minors of leading diagonal of A

$S_3 = |A|$

Note 2: Let A be a 2×2 matrix. Then the expansion of characteristic equation $|A-\lambda I| = 0$ becomes $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 =$ sum of diagonal values of A and $S_2 = |A|$.

Example: Find the characteristic equation of $A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$.

The characteristic equation of A is $|A - \lambda I| = 0$.

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of diagonal values of A $= 5 - 2 + 5 = 8$

$S_2 =$ sum of minors of leading diagonal of A

$$= \begin{vmatrix} -2 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 0 \\ 0 & -2 \end{vmatrix} = -10 + 24 - 10 = 4$$

$S_3 = |A| = 5(-10) + 1(2) = -48$

$\therefore$ the characteristic equation is $\lambda^3 - 8\lambda^2 + 4\lambda + 48 = 0$

Solved Problems

1. Show that the matrix $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

$$A \times A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$= I$

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Hence A is orthogonal

2. Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} \mathbf{2} & \mathbf{4} \\ \mathbf{1} & \mathbf{-1} \end{pmatrix}$.

The characteristic equation is $|A - \lambda I| = 0$.

$$\text{i.e.} \quad \lambda^2 - S_1\lambda + S_2 = 0 \quad \text{where}$$

$S_1 =$ sum of diagonal values of A = 2 – 1 = 1 and

$S_2 = |A| = -2 - 4 = -6.$

i.e. $\lambda^2 - \lambda - 6 = 0$

i.e. $(\lambda + 2)(\lambda - 3) = 0$

i.e. $\lambda = -2,\ \lambda = 3$ are the eigenvalues.

Consider the equation $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$\text{i.e.} \ (2 - \lambda)x + 4y = 0$$

$$x + (-1 - \lambda)y = 0$$

When $\lambda = -2$, the equations becomes

$4x + 4y = 0$

$x + y = 0$

i.e. $x = -y$

$\therefore$ when $y = 1,\ x = -1$

$\therefore X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is the eigenvector.

When $\lambda = 3$, the equations becomes

$-x + 4y = 0$

$x - 4y = 0$

i.e. $x = 4y$

$\therefore$ when $y = 1,\ x = 4$

$\therefore X_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is the eigenvector.

3. Find all the eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$.

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of diagonal values of $A = 2 + 1 - 3 = 0$

$S_2 =$ sum of minors of leading diagonal of A

$$= \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -7 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (-3-2)+(-6-0)+(2-4)$$

$$= -5-6-2$$

$$= -13$$

$S_3 = |A|$

$$= \begin{vmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{vmatrix}$$

$$= 2(-3-2)-2(-6+7)+0(4+7)$$

$$= -10-2$$

$$= -12$$

$\therefore$ the characteristic equation is $\lambda^3 - 0\lambda^2 - 13\lambda + 12 = 0$

By inspection $\lambda = 1$ is a root, since $= 1 - 13 + 12 = 0$.

By synthetic division, we have

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -13 & 12 \\ & 0 & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

$$\lambda^2 + \lambda - 12 = 0$$

$$(\lambda - 3)(\lambda + 4) = 0$$

$$\lambda = 3,\ -4$$

$\therefore \lambda = 1, 3, -4$ *are the eigenvalues*

Consider the equations $(A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$(2-\lambda)x + 2y + 0z = 0$$

$$2x + (1-\lambda)y + z = 0$$

$$-7x + 2y + (-3-\lambda)z = 0$$

Case (i) When $\lambda = 1$ the equations $(A - \lambda I)X = 0$ becomes

$$x + 2y + 0z = 0$$

$$2x + 0y + z = 0$$

$$-7x + 2y - 4z = 0$$

Solving the first and second equations, we have

$$\frac{x}{2-0} = \frac{y}{0-1} = \frac{z}{0-4}$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{-4}$$

Case (ii) When $\lambda = 3$ the equations $(A - \lambda I)X = 0$ becomes

$$-x + 2y + 0z = 0$$

$$2x - 2y + z = 0$$

$$-7x + 2y - 6z = 0$$

Solving the first and second equations, we have

$$\frac{x}{2-0} = \frac{y}{0+1} = \frac{z}{2-4}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$$\therefore\ X_1=\begin{pmatrix}-2\\1\\4\end{pmatrix}\text{ is an eigenvector.}$$

$$\therefore\ X_2=\begin{pmatrix}2\\1\\-2\end{pmatrix}\text{ is an eigen vector.}$$

Case (iii) When $\lambda=-4$ the equations $(A-\lambda I)X=0$ becomes

$$\begin{aligned}&6x+2y+0z=0\\&2x+5y+z=0\\&-7x+2y+z=0\end{aligned}$$

Solving the first and second equations, we have

$$\frac{x}{2-0}=\frac{y}{0-6}=\frac{z}{30-4}$$

$$\frac{x}{2}=\frac{y}{-6}=\frac{z}{26}$$

$$\therefore\ X_3=\begin{pmatrix}1\\-3\\13\end{pmatrix}\text{ is an eigen vector.}$$

4. Find all the eigenvalues and eigenvectors of $A=\begin{pmatrix}5&0&1\\0&-2&0\\1&0&5\end{pmatrix}$**.**

The characteristic equation of A is $|A-\lambda I|=0$

i.e. $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$ where

S_1 = sum of diagonal values of A $=5-2+5=8$

S_2 = sum of minors of leading diagonal of A

$$=\begin{vmatrix}-2&0\\0&5\end{vmatrix}+\begin{vmatrix}5&1\\1&5\end{vmatrix}+\begin{vmatrix}5&0\\0&-2\end{vmatrix}=-10+24-10=4$$

$S_3=|A|=5(-10)+1(2)=-48$

$$\therefore\ \lambda^3-8\lambda^2+4\lambda+48=0$$

By inspection $\lambda=-2$ is a root. *since* $(-2)^3-8(-2)^2+4(-2)+48=-8-32-8+48=0$

By synthetic division, we have

$$\begin{array}{r|rrrr} -2 & 1 & -8 & 4 & 48\\ & 0 & -2 & 24 & -48\\ \hline & 1 & -10 & 48 & 0\end{array}$$

$\lambda^2-10\lambda+24=0$

$(\lambda-6)(\lambda-4)=0$

$\lambda=4,6$

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$\therefore$ The eigenvalues of A are $\lambda = -2, 4, 6$

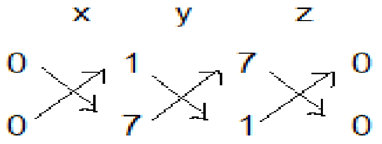
Consider the equation $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{aligned} &(5-\lambda)x + 0y + z = 0 \\ &0x + (-2-\lambda)y + 0z = 0 \\ &x + 0y + (5-\lambda)z = 0 \end{aligned}$$

Case (i): When $\lambda = -2$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$\begin{aligned} &7x + z = 0 \\ &(0)y = 0 \\ &x + 7z = 0 \end{aligned}$$

By cross multiplication rule



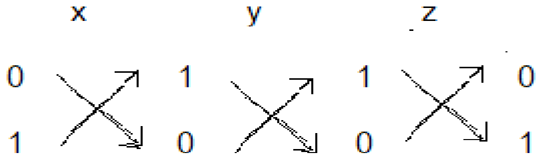
Solving first and third equation, we get $\frac{x}{0} = \frac{y}{-48} = \frac{z}{0}$ i.e. $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$

$\therefore$ the eigenvector is $X_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Case (ii): When $\lambda = 4$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

By cross multiplication rule

$$\begin{aligned} &x + z = 0 \\ &y = 0 \\ &x + z = 0 \end{aligned}$$



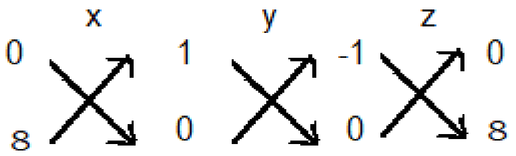
Solving first and second equation, we get $\frac{x}{0-1} = \frac{y}{0-0} = \frac{z}{1-0}$

$\therefore$ the eigenvector is $X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Case (iii): When $\lambda = 6$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$\begin{aligned} &-x + z = 0 \\ &8y = 0 \\ &x - z = 0 \end{aligned}$$

By cross multiplication rule,



Solving first and second equation, we get $\frac{x}{-8}=\frac{y}{0}=\frac{z}{-8}$ i.e. $\frac{x}{1}=\frac{y}{0}=\frac{z}{1}$

$\therefore$ the eigenvector is $X_3=\begin{pmatrix}1\\0\\1\end{pmatrix}$

5. Find all the eigenvalues and eigenvectors of $A=\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}$.

The characteristic equation of A is $|A-\lambda I|=0$

i.e. $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$ where

$S_1=$ sum of diagonal values of A $=4+4+4=12$

$S_2=$ sum of minors of leading diagonal of A

$$=\begin{vmatrix}4&1\\1&4\end{vmatrix}+\begin{vmatrix}4&1\\1&4\end{vmatrix}+\begin{vmatrix}4&1\\1&4\end{vmatrix}=15+15+15=45$$

$S_3=|A|=4(16-1)-1(4-1)+1(1-4)=60-3-3=54$

$\therefore$ $\lambda^3-12\lambda^2+45\lambda-54=0$

By inspection $\lambda=3$ is a root. since $(3)^3-12(3)^2+45(3)-54=27-108+135-54=0$

By synthetic division, we have

3	1	–12	45	–54
	0	3	–27	54
	1	–9	18	0

$\lambda^2-9\lambda+18=0$

$(\lambda-3)(\lambda-6)=0$

$\lambda=3,6$

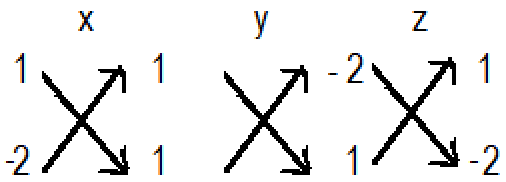
$\therefore$ The eigenvalues of A are $\lambda=6,3,3$

Consider the equation $(A-\lambda I)X=0$ where $X=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$(4-\lambda)x+y+z=0$$
$$x+(4-\lambda)y+z=0$$
$$x+y+(4-\lambda)z=0$$

Case (i): When $\lambda=6$, the simultaneous equations $(A-\lambda I)X=0$ becomes

$$-2x+y+z=0$$
$$x-2y+z=0$$
$$x+y-2z=0$$



Solving first and second equation, we get $\frac{x}{3}=\frac{y}{3}=\frac{z}{3}$ i.e. $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$

$\therefore$ the eigenvector is $X_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$

Case (ii): When $\lambda=3$, the simultaneous equations $(A-\lambda I)X=0$ becomes

$$x+y+z=0$$
$$x+y+z=0$$
$$x+y+z=0$$

All the three equations are identical and same as to $x+y+z=0$

$\therefore$ two unknowns may be treated as parameters. Taking $x=1,\ y=0$, we get $z=-1$

Also taking $x=0, y=1,$ we get $z=-1$

$\therefore$ the eigenvectors are $X_2=\begin{pmatrix}1\\0\\-1\end{pmatrix}$ and $X_3=\begin{pmatrix}0\\1\\-1\end{pmatrix}$

Note: Though two of the eigenvalues are equal, the eigenvectors are linearly independent. It can be seen from the fact that $k_1X_1+k_2X_2+k_3X_3=0\ when\ k_1=k_2=k_3=0$

6. Find all the eigenvalues and eigenvectors of $A=\begin{pmatrix}3&10&5\\-2&-3&-4\\3&5&7\end{pmatrix}$.

The characteristic equation of A is $|A-\lambda I|=0$

i.e. $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$ where

S_1 = sum of diagonal values of $A=3-3+7=7$

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$S_2 =$ sum of minors of leading diagonal of A

$$= \begin{vmatrix} -3 & -4 \\ 5 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 10 \\ -2 & -3 \end{vmatrix}$$

$$= (-21+20)+(21-15)+(-9+20)$$

$$= -1+6+11$$

$$= 16$$

$S_3 = |A|$

$$= \begin{vmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{vmatrix}$$

$$= 3(-21+20)-10(-14+12)+5(-10+9)$$

$$= -3+20-5$$

$$= 12$$

$\therefore$ the characteristic equation is $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

By inspection $\lambda = 2$ is a root, since $2^3 - 7\times 2^2 + 16\times 2 - 12 = 8 - 28 + 32 - 12 = 0$.

By synthetic division, we have

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-3)(\lambda-2) = 0$$

$$\lambda = 2, 3$$

$$\therefore \lambda = 3,\ 2,\ 2 \textit{ are the eigenvalues}$$

2	**1**	**-7**	**16**	**-12**
	0	**2**	**-10**	**12**
	1	**-5**	**6**	**0**

Consider the equations $(A-\lambda I)X = 0$

$$\begin{pmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$(3-\lambda)x + 10y + 5z = 0$$

$$-2x + (-3-\lambda)y - 4z = 0$$

$$3x + 5y + (7-\lambda)z = 0$$

Case (i) When $\lambda = 3$ the equations $(A-\lambda I)X = 0$ becomes

$$0x + 10y + 5z = 0$$

$$-2x - 6y - 4z = 0$$

$$3x + 5y + 4z = 0$$

Solving the first and second equations, we have

$$\frac{x}{-40+30} = \frac{y}{-10-0} = \frac{z}{0+20}$$

$$\frac{x}{-10} = \frac{y}{-10} = \frac{z}{20}$$

$\therefore\ X_1 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ is an eigenvector.

Case (ii) When $\lambda = 2$ the equations $(A-\lambda I)X = 0$ becomes

$$x + 10y + 5z = 0$$

$$-2x - 5y - 4z = 0$$

$$3x + 5y + 5z = 0$$

Solving the first and second equations, we have

$$\frac{x}{-40+25} = \frac{y}{-10+4} = \frac{z}{-5+20}$$

$$\frac{x}{-15} = \frac{y}{-6} = \frac{z}{15}$$

$\therefore\ X_2 = X_3 = \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}$ are the eigenvectors, as all the simultaneous equation are distinct.

Note: Here two eigenvalues are equal and the eigen vectors are linearly dependent.

7. Find all the eigenvalues and eigenvectors of $A=\begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}$**.**

The characteristic equation of A is $|A-\lambda I|=0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of diagonal values of A $=6+3+3=12$

$S_2 =$ sum of minors of leading diagonal of A

$$=\begin{vmatrix}3 & -1\\ -1 & 3\end{vmatrix}+\begin{vmatrix}6 & 2\\ 2 & 3\end{vmatrix}+\begin{vmatrix}6 & -2\\ -2 & 3\end{vmatrix}=8+14+14=36$$

$S_3 = |A| = 6(9-1)+2(-6+2)+2(2-6)=48-8-8=32$

$\therefore\ \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

By inspection $\lambda = 2$ is a root. *since* $(2)^3 - 12(2)^2 + 36(2) - 32 = 8 - 48 + 72 - 32 = 0$

By synthetic division, we have

$$\begin{array}{r|rrr|r} 2 & 1 & -12 & 36 & -32\\ & 0 & 2 & -20 & 32\\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$\lambda^2 - 10\lambda + 16 = 0$

$(\lambda-2)(\lambda-8)=0$

$\lambda = 2, 8$

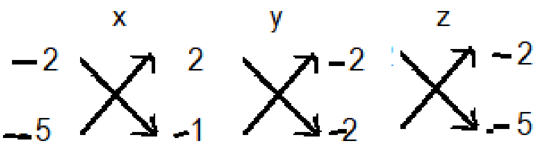
$\therefore$ The eigenvalues of A are $\lambda = 8, 2, 2$

Consider the equation $(A-\lambda I)X = 0$ where $X=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

$$(6-\lambda)x - 2y + 2z = 0$$
$$-2x + (3-\lambda)y - z = 0$$
$$2x - y + (3-\lambda)z = 0$$

Case (i): When $\lambda = 8$, the simultaneous equations $(A-\lambda I)X = 0$ becomes

$$-2x - 2y + 2z = 0$$
$$-2x - 5y - z = 0$$
$$2x - y - 5z = 0$$



Solving first and second equation, we get

$$\frac{x}{12}=\frac{y}{-6}=\frac{z}{6} \quad \text{i.e.} \quad \frac{x}{2}=\frac{y}{-1}=\frac{z}{1}$$

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$\therefore$ the eigen vector is $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Case (ii): When $\lambda = 2$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$4x - 2y + 2z = 0$$
$$-2x + y - z = 0$$
$$2x - y + z = 0$$

All the three equations are identical and same as to $2x - y + z = 0$

$\therefore$ two unknowns may be treated as parameters. Taking $x = 1,\ y = 2$, we get $z = 0$

$\therefore$ the eigenvector corresponding to $\lambda = 2$ is $X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Assume that $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the third eigenvector corresponding to $\lambda = 2$ such that

$$X_2 . X_3^T = 0 \quad \text{and} \quad X_1 . X_3^T = 0$$

$$\therefore \quad a + 2b = 0 \quad \text{and} \quad 2a - b + c = 0$$

Solving the equations, we get $\dfrac{a}{2} = \dfrac{b}{-1} = \dfrac{c}{-5}$ and hence $X_3 = \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix}$

Problems based on properties of eigenvalues and eigen vectors

1. If $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$, what are the eigenvalues of A^{-1}?

Since A is an upper triangular matrix, the eigenvalues of A are the diagonal elements, say 3, 2, 5.

$\therefore$ the eigenvalues of A^{-1} are $\dfrac{1}{3}, \dfrac{1}{2}, \dfrac{1}{5}$.

2. Find a and b such that the matrix $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ has 3 and -2 as its eigenvalues.

By property of eigenvalues,

$$\lambda_1 + \lambda_2 = a + b \qquad \text{and} \qquad \lambda_1 \lambda_2 = ab - 4$$
$$3 - 2 = a + b \qquad\qquad -6 = ab - 4$$
$$a + b = 1 \qquad\qquad ab = -2$$

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$$\text{We know that } (a-b)^2 = (a+b)^2 - 4ab$$

$$= 1+8 = 9$$

$$a-b = \pm 3$$

Solving $a+b=1$ and $a-b=3$, we get $a=2,\ b=-1$

3. **If the sum of two eigenvalues and trace of a 3×3 matrix A are equal, find |A|.**

Given $\lambda_1+\lambda_2$ = $\lambda_1+\lambda_2+\lambda_3$ and hence $\lambda_3=0$

$\therefore\ \lambda_1\times\lambda_2\times\lambda_3 = |A|$

i.e. $|A|=0$

4. **Two eigenvalues of $A=\begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}$ are equal and they are double the third. Find the eigenvalues of A^2.**

By property, $\lambda_1+\lambda_2+\lambda_3$ = 4 + 3 – 2 = 5

Given that $\lambda_1=\lambda_2$ and $\lambda_2 = 2\lambda_3$

$$\therefore\ \lambda_2+\lambda_2+\frac{\lambda_2}{2} = 5$$

$$\frac{5}{2}\lambda_2 = 5$$

$\lambda_2 = 2$ and hence $\lambda_1 = 2$, $\lambda_3 = 1$

5. **Find the eigenvalue of $A=\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ whose eigenvector is $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$.**

We know that $AX=\lambda X$

$$\begin{pmatrix} 4+0+2 \\ 0+0+0 \\ -2+0-4 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ 0 \\ -2\lambda \end{pmatrix}$$

i.e. $2\lambda = 6$ and $-2\lambda = -6$

$\therefore$ the eigenvalue of A is $\lambda = 3$.

6. **If $A^{-1} = \frac{1}{24}\begin{pmatrix} 2 & 4 \\ -2 & 8 \end{pmatrix}$, find the eigenvalues of A.**

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We know that $A^{-1} = \frac{Adj(A)}{|A|}$. Given that $A^{-1} = \frac{1}{24}\begin{pmatrix} 2 & 4 \\ -2 & 8 \end{pmatrix}$

Comparing these two, we have $Adj(A) = \begin{pmatrix} 2 & 4 \\ -2 & 8 \end{pmatrix}$ and hence $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$

The characteristic equation of A is $\begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$

$$(8-\lambda)(2-\lambda)+8=0$$
$$\lambda^2 - 10\lambda + 24 = 0$$
$$(\lambda-6)(\lambda-4)=0$$

$\therefore$ the eigenvalues of A are $\lambda = 4, 6$.

7. One of the eigenvalue of $A = \begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{pmatrix}$ is -9. Find the other two eigenvalues.

Let $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A.

Then $\lambda_1 + \lambda_2 + \lambda_3$ = sum of diagonal elements of A = -9

$-9 + \lambda_2 + \lambda_3 = -9$

$\lambda_2 = -\lambda_3$ ……..(1)

Also $\lambda_1 \lambda_2 \lambda_3 = |A| = 7(64-1) - 4(-32+4) - 4(-4+32) = 441$

$\frac{\lambda_2\lambda_3}{\lambda_1} = \frac{441}{-9} = -49$ $-\lambda_3^2 = -49$ $\lambda_3 = \pm 7$

$\therefore$ $\lambda_2 = 7$, $\lambda_3 = -7$

8. If the eigenvalues of $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ are 1, 6. Find the eigenvalues of A^{-1} and A^2

Since 0 is not an eigen value of A, A is non singular matrix and hence A^{-1} exists.

Eigenvalues of A^{-1} are $\frac{1}{1}, \frac{1}{6}$

Eigenvalues of A^2 are $1^2, 6^2$

9. Find the eigenvalues of A^3 if $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$

Since A is a lower triangular matrix, the eigenvalues are the diagonal elements. i.e. $\lambda = 1, 3, 6$

$\therefore$ the eigenvalues of A^3 are $1^3, 3^3, 6^3$.

10. Find the sum and product of eigenvalues of the matrix $A=\begin{pmatrix}1&1&3\\1&5&1\\3&1&1\end{pmatrix}$ without actually finding the eigenvalues.

By a property, sum of all eigenvalues of A = Trace of A

= 1 + 5 + 1

= 7

Product of eigenvalue = $|A|$

= 1 (5 – 1) –1 (1 – 3) + 3 (1 – 15)

= 4 + 2 – 42

= – 36

11. Find the eigenvalues of $2A-I$ if the the matrix $A=\begin{pmatrix}-4&1\\3&-2\end{pmatrix}$

The characteristic equation is $\begin{vmatrix}-4-\lambda&1\\3&-2-\lambda\end{vmatrix}=0$

$$8+4\lambda+2\lambda+\lambda^2-3=0$$

$$\lambda^2+6\lambda+5=0$$

$$(\lambda+1)(\lambda+5)=0$$

$\therefore$ the eigen values of A are $\lambda=-1,-5$ and hence the eigen values of 2A are $-2,-10$

The eigen values of the identity matrix I are 1, 1

$\therefore$ the eigen values of 2A–I are –2–1, –10–1 i.e. –3, –11

12. If 2, –1, 3 are the eigen values of a matrix A, then find the eigen values of A^2-2I.

Since 2, –1, 3 are the eigen values of a matrix A, the eigen values of A^2 are $2^2,\ (-1)^2\ ,3^2$

Therefore the eigen values of A^2-2I are $2^2-2,\ (-1)^2-2\ ,3^2-2$ *i.e.* $2,\ -1,\ 7$.

13. If the product of two eigenvalues of $A=\begin{pmatrix}1&0&0\\0&3&-1\\0&-1&3\end{pmatrix}$ is 2, find the third eigenvalue.

Let x,y,z be the eigenvalues of A. By properties of eigenvalues,

$x+y+z=1+3+3=7$ ....(1) and $xyz=|A|=8$ ....(2)

Given that $xy=2$ and hence $z=4$. Also $y=\dfrac{2}{x}$ ....(3)

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From (1), $x+\frac{2}{x}+4=7$

$$x^2+4x+2=7x$$

$$x^2-3x+2=0$$

$$(x-1)(x-2)=0$$

x = 1, 2

$\therefore\ x=1, y=2, z=4$ are the eigenvalues

14. Two eigenvalues of the matrix $A=\begin{pmatrix}2&2&1\\1&3&1\\1&2&2\end{pmatrix}$ are equal to 1 each. Find the eigen values of A^3.

Given that 1, 1, λ are the eigenvalues of A.

By a property, $1+1+\lambda=2+3+2 \ \Rightarrow\ \lambda=5$

$\therefore$ eigenvalues of A are 1, 1, 5 and hence eigenvalues of A^3 are $1^3, 1^3, 5^3$.

15. One of the eigenvalues of $A=\begin{pmatrix}1&1&3\\1&5&1\\3&1&1\end{pmatrix}$ is 6. Find the other two eigenvalues.

Let $x, y, 6$ are the eigenvalues of A.

By properties of eigenvalues, $x+y+6=1+5+1$ and $6xy=|A|$

i.e., $x+y=1$ .....(1) and $xy=-6$ .....(2)

From (2), $y=-\frac{6}{x}$. Using this in (1), we get $x-\frac{6}{x}=1$

$$x^2-x-6=0$$

$$(x+2)(x-3)=0$$

$$x=-2,3$$

$\therefore$ the other two eigenvalues are –2, 3.

16. Determine which of the following vectors are eigenvector of $A=\begin{pmatrix}4&2\\5&1\end{pmatrix}$. Also find the corresponding eigenvalue if any: $X_1=\begin{pmatrix}5\\-2\end{pmatrix}$, $X_2=\begin{pmatrix}-2\\5\end{pmatrix}$

Consider $AX_2=\begin{pmatrix}4&2\\5&1\end{pmatrix}\begin{pmatrix}-2\\5\end{pmatrix}=\begin{pmatrix}2\\-5\end{pmatrix}=-1\begin{pmatrix}-2\\5\end{pmatrix}=\lambda X_2$

$\therefore\ X_2$ is the eigenvector corresponding to the eigenvalue $\lambda=-1$.

But $AX_1\neq\lambda X_1$ and hence X_1 is not an eigenvector.

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17. If -1 and 2 are the eigen values of $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & 1 & 0 \end{pmatrix}$, find the value of 'a'.

We know that, Sum of eigen values = sum of diagonal elements of the matrix

$$\lambda_1+\lambda_2+\lambda_3=0+0+0$$

$$-1+2+\lambda_3=0$$

$$\lambda_3=-1$$

Product of the eigen value = determinant value of the matrix

$$\lambda_1\times\lambda_2\times\lambda_3=\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & 1 & 0 \end{vmatrix}$$

$$(-1)\times(2)\times(-1)=-1(0-a)+1(1-0)$$

$$2=a+1$$

$$a=1$$

18. If the sum and product of eigen values of a 2×2 matrix A are 2 and –3 respectively, compute the eigen values of A.

Let λ_1,λ_2 are the eigen values of the matrix A. Given that

$$\lambda_1+\lambda_2=2\ldots(1)$$
$$\lambda_1\ \lambda_2=-3\ldots(2)$$

From (1), $\lambda_2=2-\lambda_1$. From (2), $\lambda_1\left(2-\lambda_1\right)=-3$

$$-\lambda_1^2+2\lambda_1+3=0$$

$$\lambda_1^2-2\lambda_1-3=0$$

$$\left(\lambda_1+1\right)\left(\lambda_1-3\right)=0$$

$$\lambda_1=-1\ \ or\ \ \lambda_1=3$$

$$\therefore\ If\ \ \lambda_1=-1,\ \lambda_2=2-(-1)=3$$

19. If λ is an eigenvalue of an orthogonal matrix A, show that $\frac{1}{\lambda}$ is also an eigenvalue of A.

Let λ be the eigenvalue of an orthogonal matrix A. Since A is orthogonal, then $A^T=A^{-1}$.

By a property, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Therefore and $\frac{1}{\lambda}$ is an eigenvalue of A^T.

Also, by a property, A and A^T have same eigenvalues.

Therefore and $\frac{1}{\lambda}$ is an eigenvalue of A.

20. If λ is an eigenvalue of a matrix A, show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}..

If λ is an eigenvalue of a matrix A, then $AX = \lambda X$.

Premultiply by A^{-1}, we have $A^{-1}.AX = \lambda A^{-1}.X$

$$X = \lambda A^{-1}.X$$

$$\frac{1}{\lambda}X = A^{-1}.X$$

Therefore $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

21. Prove that $x^2 - y^2 + 4z^2 + 4xy + 2yz + 6xz$ is indefinite.

The matrix of the quadratic form is $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{pmatrix}$

Here $D_1 = |1| = 1, +ve$ $\qquad$ $D_2 = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5, -ve$

$$D_3 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 1(-4-1) - 2(8-3) + 3(2+3) = -5 - 10 + 15 = 0$$

Since $D_1 = +ve, D_2 = -ve, D_3 = 0,$ the quadratic form is indefinite.

Exercise

1. Find the eigenvalues and eigenvectors of the following matrices:

(i) $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$ (iv) $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

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2. Find all the eigen values and eigen vectors of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

3. Find the eigenvalues and eigen vectors of the matrix $A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$.

4. Show that the matrix $\mathrm{A} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is orthogonal.

5. Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

6. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$, what are the eigen values of A^2?

7. Find a and b such that the matrix $\begin{pmatrix} \mathrm{a} & -1 \\ 2 & \mathrm{b} \end{pmatrix}$ has 3 and 2 as its eigen values.

8. If the sum of two eigen values and trace of a 3×3 matrix A are equal, find the product of all its eigen values.

9. Two eigen values of $A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$ are equal and the third is double of them. Find the eigen values of A^2.

10. Find the eigen value of $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ whose eigen vector is $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

11. If $\mathrm{A}^{-1} = \frac{1}{6}\begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}$, find the eigen values of A.

12. One of the eigen value of $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 8. Find the other two eigen values.

13. If the eigen values of $\mathrm{A} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$ are 1, −1. Find the eigen values of A^{-1} and A^2

14. Find the eigen values of A^3 if $\mathrm{A} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

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15. Find the sum and product of eigen values of the matrix $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ without actually finding the eigen values.

16. Find the eigen values of $2A + 3I$ if the the matrix $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

17. If the product of two eigen values of $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is 1, find the third eigen value.

18. Two eigen values of the matrix $A = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$ are equal to 2 each. Find the eigen values of A^3.

19. One of the eigen values of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ is 4. Find the other two eigen values.

20. Determine which of the following vectors are eigen vector of $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

21. If 1 and 2 are the eigen values of $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{pmatrix}$, find the value of 'a'.

Cayley Hamilton Theorem

Statement: Every square matrix satisfies its own characteristic equation.

i.e. If $a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots + a_{n-1}\lambda + a_n = 0$ is the characteristic equation of a square matrix A of order n, then $a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots + a_{n-1}A + a_nI = O$ ....(1)

Note:

1. The RHS of (1) is a null matrix of order n.
2. If A is non singular, A^{-1} can be obtained using this theorem.
3. Any positive integral power of A can be expressed as a linear combination those of lower degree.
4. $adj(A) = -a_0A^{n-1}a_1A^{n-2} - a_2A^{n-3} - \ldots\ldots\ldots - a_{n-1}I$

Solved Problems

1. If 2, 3 are eigen values of a square matrix A of order 2, express A^2 in terms of A and I.

Since 2, 3 are the eigen values, the characteristic equation is of the form

$$(\lambda-2)(\lambda-3)=0$$

$$\lambda^2-5\lambda+6=0$$

$\therefore$ By Cayley Hamilton theorem, $A^2-5A+6I=0$. $\therefore$ $A^2=5A-6I$

2. If 1, 2, 3 are eigenvalues of a square matrix A of order 3, express A^3 in terms of lower powers of A and I.

Since 1, 2, 3 are the eigenvalues, the characteristic equation is of the form

$$(\lambda-1)(\lambda-2)(\lambda-3)=0$$

$$(\lambda^2-3\lambda+2)(\lambda-3)=0$$

$$(\lambda^3-3\lambda^2+2\lambda-3\lambda^2+9\lambda-6)=0$$

$$\lambda^3-6\lambda^2+11\lambda-6=0$$

$\therefore$ By Cayley Hamilton theorem, $A^3-6A^2+11A-6I=0$

Therefore $A^3=6A^2-11A+6I$

3. Verify Cayley Hamilton Theorem for $A=\begin{pmatrix}0 & 2\\ 4 & 0\end{pmatrix}$. Also find A^4.

The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix}0-\lambda & 2\\ 4 & 0-\lambda\end{vmatrix}=0$$

$$\lambda^2-8=0$$

$\therefore$ By Cayley Hamilton Theorem, $A^2-8I=0$ ……..(1)

$$A^2=A\times A \;=\begin{pmatrix}0 & 2\\ 4 & 0\end{pmatrix}\times\begin{pmatrix}0 & 2\\ 4 & 0\end{pmatrix}\;=\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}$$

$$A^2-8I=\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}-8\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\;=\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}-\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}\;=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Hence the theorem is verified.

Multiply (1) by A^2, we get $A^4-8A^2=0$

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$$A^4 = 8A^2$$

$$= 8\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 64 & 0 \\ 0 & 64 \end{pmatrix}$$

4. Using Cayley Hamilton Theorem, find the inverse of $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}$

The characteristic equation of A is $| A - \lambda I |= 0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of diagonal values of A $= 4 + 3 - 3 = 4$

$S_2 =$ sum of minors of leading diagonal of A

$$= \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix} = -1 - 6 + 6 = -1$$

$S_3 =| A |= 4(-9+8) - 6(-3+2) + 6(-4+3) = -4 + 6 - 6 = -4$

$\therefore$ $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$\therefore$ By Cayley Hamilton theorem, we have $A^3 - 4A^2 - A + 4I = 0$ ………..(1)

Pre multiply equation (1) by A^{-1}, we get $A^2 - 4A - I + 4A^{-1} = 0$

$$A^{-1} = \frac{1}{4}(-A^2 + 4A + I)$$

$$A^2 = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix} \times \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix} = \begin{pmatrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ -5 & -6 & -5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{4}\left\{\begin{pmatrix} -16 & -18 & -18 \\ -5 & -7 & -6 \\ -5 & -6 & -5 \end{pmatrix} + \begin{pmatrix} 16 & 24 & 24 \\ 4 & 12 & 8 \\ -4 & -16 & -12 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} = \frac{1}{4}\begin{pmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{pmatrix}$$

5. Apply Cayley Hamilton Theorem to find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

The characteristic equation of A is $| A - \lambda I |= 0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of diagonal values of $A = 1 + 4 + 6 = 11$

$S_2 =$ sum of minors of leading diagonal of A

$$= \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= (24-25)+(6-9)+(4-4)$$

$$= -1-3$$

$$= -4$$

$S_3 = |A|$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}$$

$$= 1(24-25)-2(12-15)+3(10-12)$$

$$= -1+6-6$$

$$= -1$$

$\therefore$ the characteristic equation is $\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$

$\therefore$ By Cayley Hamilton theorem, we have $A^3 - 11A^2 - 4A + I = 0$

Pre multiply equation (1) by A^{-1}, we get $A^2 - 11A - 4I + A^{-1} = 0$

$$A^{-1} = -A^2 + 11A + 4I$$

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix}$$

$$A^{-1} = -\begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} + 11\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

6. **Use Caley Hamilton Theorem to find** $A^4 - 4A^3 - 5A^2 + A + 2I$ if $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$\therefore$ By Caley Hamilton Theorem, $A^2 - 4A - 5I = 0$ ………….(1)

$A^4 - 4A^3 - 5A^2 + A + 2I$ **=** $A^2(A^2 - 4A - 5I) + A + 2I$

$= (A + 2I)$ { Using (1) }

$$= \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 \\ 6 & 5 \end{pmatrix}$$

7. Given $A=\begin{pmatrix}2 & -1\\ 1 & 3\end{pmatrix}$, express $A^4-4A^3-A^2+2A-5I$ as a linear polynomial in A and hence evaluate it.

The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix}2-\lambda & -1\\ 1 & 3-\lambda\end{vmatrix}=0$$

$$(2-\lambda)(3-\lambda)+1=0$$

$$\lambda^2-5\lambda+7=0$$

$\therefore$ By Cayley Hamilton Theorem,

$A^2-5A+7I=0$ ………….(1)

Dividing $A^4-4A^3-A^2+2A-5I$ by (1), we get

$A^4-4A^3-A^2+2A-5I$

$$= (A^2-5A+7I)\ (A^2+A-3I) + (-20A+16I)$$

$$=(-20A+16I) \qquad \{\text{Using (1)}\}$$

$$=-20\begin{pmatrix}2 & -1\\ 1 & 3\end{pmatrix}+16\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-24 & 20\\ -20 & -44\end{pmatrix}$$

$$\begin{array}{r|l}
 & A^2+A-3I \\ \hline
A^2-5A+7I & A^4-4A^3-A^2+2A-5I \\
 & A^4-5A^3+7A^2 \\ \hline
 & A^3-8A^2+2A-5I \\
 & A^3-5A^2+7A \\ \hline
 & -3A^2-5A-5I \\
 & -3A^2+15A-21I \\ \hline
 & -20A+16I
\end{array}$$

8. Verify Caley Hamilton Theorem for $A=\begin{pmatrix}1 & 1 & 3\\ 1 & 3 & -3\\ -2 & -4 & -4\end{pmatrix}$. Also find adj(A) and A^{-1}.

The characteristic equation of A is $|A-\lambda I|=0$

i.e. $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$ where

$S_1=$ sum of diagonal values of A $=1+3-4=0$

$S_2=$ sum of minors of leading diagonal of A

$$=\begin{vmatrix}3 & -3\\ -4 & -4\end{vmatrix}+\begin{vmatrix}1 & 3\\ -2 & -4\end{vmatrix}+\begin{vmatrix}1 & 1\\ 1 & 3\end{vmatrix}=-24+2+2=-20$$

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$$S_3 = |A| = 1(-12-12) - 1(-4-6) + 3(-4+6) = -24+10+6 = -8$$

$\therefore\ \lambda^3 - 20\lambda + 8 = 0$

$\therefore$ By Cayley Hamilton theorem, we have to prove $A^3 - 20A + 8I = 0$ ……….(1)

$$A^2 = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix} = \begin{pmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix} = \begin{pmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{pmatrix}$$

$$A^3 - 20A + 8I = \begin{pmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{pmatrix} - 20\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the theorem is verified.

From (1), $Adj\ A = -A^2 + 20I$

$$= \begin{pmatrix} 4 & 8 & 12 \\ -10 & -22 & -6 \\ -2 & -2 & -22 \end{pmatrix} + \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix} = \begin{pmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{pmatrix}$$

Pre multiply (1) by A^{-1}, we get $A^2 - 20I + 8A^{-1} = 0$

$$A^{-1} = \frac{1}{8}(-A^2 + 20I) = \frac{1}{8}\begin{pmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{pmatrix}$$

9. Verify Cayley Hamilton Theorem for $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$**. Hence compute** A^{-1}

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = sum of diagonal values of A $= 2+2+2 = 6$

S_2 = sum of minors of leading diagonal of A

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = (4-1) + (4-1) + (4-1) = 9$$

$$S_3 = |A| = 2(4-1) + 1(-2+1) + 1(1-2) = 6-1-1 = 4$$

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$\therefore\ \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$\therefore$ By Cayley Hamilton theorem, we have to prove $A^3 - 6A^2 + 9A - 4I = O$ …….(1)

$$A^2 = A \times A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{pmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the theorem is verified.

Pre multiply (1) by A^{-1}, we get $A^2 - 6A + 9I - 4A^{-1} = 0$

$$A^{-1} = \frac{1}{4}\left\{A^2 - 6A + 9I\right\} = \frac{1}{4}\left\{\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$$= \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

10. Using Cayley Hamilton Theorem find A^{-1} and $A^3 + A^6$, if $A = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$.

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 \\ 5 & -2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-2-\lambda) + 5 = 0$$

$$\lambda^2 + 1 = 0$$

$\therefore$ By Cayley Hamilton Theorem, $A^2 + I = 0$ ……..(1)

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Pre multiply equation (1) by A^{-1}, we get $A + A^{-1} = 0$

$$A^{-1} = -A$$

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix}$$

Multiply equation (1) by A, we get $A^3 + A = 0$

$$A^3 = -A$$

$$A^6 = A^2$$

$$A^3 + A^6 = A^2 - A$$

$$= (A^2 + I) - A - I$$

$$= -A - I \qquad \{\text{ using (1) }\}$$

$$= \begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 \\ -5 & 1 \end{pmatrix}$$

11. Using Cayley Hamilton Theorem find A^n if $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and hence find A^3.

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda + 1)(\lambda - 5) = 0$$

$\therefore$ the eigen values are $\lambda = -1, 5$

$\therefore$ By Cayley Hamilton theorem, we have $A^2 - 4A - 5I = 0$ ....(1)

Dividing λ^n by $\lambda^2 - 4\lambda - 5$, the quotient is $Q(\lambda)$ and the reminder is $(a\lambda + b)$, then

$\lambda^n = (\lambda^2 - 4\lambda - 5)\ Q(\lambda) + (a\lambda + b)$ ....(2)

Putting $\lambda = -1, 5$ in equation (1), we get $-a + b = (-1)^n$ and $5a + b = 5^n$

Solving, we get $a = \frac{1}{6}\left[5^n - (-1)^n\right],\ b = \frac{1}{6}\left[5^n + 5(-1)^n\right]$

Put $\lambda = A$ in equation (2), we get $A^n = aA + bI$ {using (1)}

$$= \frac{1}{6}\left[5^n - (-1)^n\right]\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + \frac{1}{6}\left[5^n + 5(-1)^n\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \frac{126}{6}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + \frac{120}{6}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 21\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + 20\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 41 & 42 \\ 84 & 83 \end{pmatrix}$$

Exercise

1. Verify Cayley Hamilton theorem for the following matrices:

   (i) $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$

2. Use Cayley Hamilton theorem to find the inverse of the following matrices:

   (i) $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ (iii) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ (iv) $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

3. Use Cayley Hamilton theorem to find $A^4 - 4A^3 - 5A^2 + A + 2I$, when $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

4. If 2, −1 are eigen values of a square matrix A of order 2, express A^2 in terms of A and I.

5. If 1, 2, −1 are eigen values of a square matrix A of order 3, express A^3 in terms of lower powers of A and I.

6. If $A^2 = I$, what do you understand by Cayley Hamilton theorem about the eigenvalues?

7. Verify Cayley Hamilton Theorem for $A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

8. Given $A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, express $A^4 - 4A^3 - A^2 + 2A - 5I$ as a linear polynomial in A and hence evaluate it.

9. Use Caley Hamilton Theorem to find $A^4 - 4A^3 - 5A^2 + A + 2I$ if $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

10. Using Caley Hamilton Theorem find A^{-1} and $A^3 + A^6$, if $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

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11. Using Caley Hamilton Theorem find A^n if A $= \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$ and hence find A^3.

Diagonalisation

Two matrices A and B are said to be **similar**, tf there exists a non singular matrix P such that $B = P^{-1}AP$ are said to be similar matrices.

If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

Let A be a matrix of order 3 and let X_1, X_2, X_3 be the eigen vectors corresponding to the eigen values λ_1, λ_2, λ_3 respectively. Then $AX_1 = \lambda_1 X_1$, $AX_2 = \lambda_2 X_2$, $AX_3 = \lambda_3 X_3$

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Let $P=[X_1 \quad X_2 \quad X_3]$. Then $AP = A\ [X_1 \quad X_2 \quad X_3]$

$$=[AX_1 \quad AX_2 \quad AX_3]$$

$$=[\lambda X_1 \quad \lambda X_2 \quad \lambda X_3]$$

$$=[X_1 \quad X_2 \quad X_3]\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}$$

$$=PD$$

Premultiply by P^{-1}, we get $P^{-1}AP=D$

Note:

1. The matrix P which diagonalises A is called modal matrix.
2. A is diagonalizable if and only if its eigen values are distinct otherwise the eigen vectors must be linearly independent.

Calculation of powers of a matrix

The diagonal form D of A is given by $D = P^{-1}AP$

$$AP=PD$$

$$A=PDP^{-1}$$

$$A^k=(PDP^{-1})(PDP^{-1})(PDP^{-1})........(PDP^{-1})\ (k\ factors)$$

$$A^k=PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)D.........(P^{-1}P)DP^{-1}$$

$$A^k=PD^kP^{-1}$$

Diagonalisation by orthogonal transformation

Let A be a real symmetric matrix. Then the eigenvectors of A will be linearly independent as well as pairwise orthogonal. Then we use normalized eigenvectors of A to form the normalized modal matrix P. Now it can be proved that P is an orthogonal matrix then $P^T=P^{-1}$.

Note:

1. Let X be an eigenvector. Divide each element of X by the square root of the sum of the squares of all the elements of X. The resulting eigenvector is the normalized eigenvector.

2. If $X=\begin{pmatrix}1\\2\end{pmatrix}$ is an eigenvector, then its normalised form is $\begin{pmatrix}\frac{1}{\sqrt{1^2+2^2}}\\ \frac{2}{\sqrt{1^2+2^2}}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix}$

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3. If $X = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is an eigenvector, then its normalised form is $\begin{pmatrix} \frac{1}{\sqrt{1^2+2^2+2^2}} \\ \frac{2}{\sqrt{1^2+2^2+2^2}} \\ \frac{2}{\sqrt{1^2+2^2+2^2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$

Working Rule:

1. Find the eigenvalues of the given matrix
2. Find the eigenvectors
3. Form the modal matrix M whose columns are the eigenvectors
4. Form the normalised modal matrix N. Now N is orthogonal and hence $N^T = N^{-1}$.
5. Then $N^{-1}AN = N^TAN = D$ where D is a diagonal matrix whose elements are the eigenvalues.

Solved Problems

1. Can we diagonalise $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$?

Obviously, the eigen values are $\lambda = 1, 1$

Consider the equation $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$
$$(1-\lambda)x + 0y = 0$$
$$0x + (1-\lambda)y = 0$$

When $\lambda = 1$, the above equations becomes,

$$(0)x = 0$$
$$(0)y = 0$$

Let one solution is $X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the other solution is $X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Also they are independent.

Hence the given matrix can be diagonalised.

2. Diagonalise $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ by orthogonal reduction and hence find A^2.

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

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$S_1 =$ sum of diagonal values of A $= 3+5+3 = 11$

$S_2 =$ sum of minors of leading diagonal of A

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} = 14+8+14 = 36$$

$S_3 = |A| = 3(15-1)+1(-3+1)+1(1-5) = 42-2-4 = 36$

$\therefore\ \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

By inspection $\lambda = 2$ is a root.

By synthetic division, we have

$\lambda^2 - 9\lambda + 18 = 0$

$(\lambda - 3)(\lambda - 6) = 0$

$\lambda = 3, 6$

$\therefore$ The eigen values of A are $\lambda = 2, 3, 6$

(Try synthetic division here)

Consider the equation $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(3-\lambda)x - y + z = 0$$
$$-x + (5-\lambda)y - z = 0$$
$$x - y + (3-\lambda)z = 0$$

Case (i): When $\lambda = 2$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$x - y + z = 0$$
$$-x + 3y - z = 0$$
$$x - y + z = 0$$

$$\begin{array}{ccccccc} & x & & y & & z & \\ -1 & & 1 & & 1 & & -1 \\ 3 & & -1 & & -1 & & 3 \end{array}$$

Solving first and second equation, we get $\dfrac{x}{-2} = \dfrac{y}{0} = \dfrac{z}{2}$ i.e. $\dfrac{x}{-1} = \dfrac{y}{0} = \dfrac{z}{1}$

$\therefore$ the eigenvector is $X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Case (ii): When $\lambda = 3$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$\begin{aligned} -y+z&=0 \\ -x+2y-z&=0 \\ x-y&=0 \end{aligned}$$

$$\boxed{\begin{array}{ccccccc} & x & & y & & z & \\ -1 & & 1 & & 0 & & -1 \\ -1 & & 0 & & 1 & & -1 \end{array}}$$

Solving first and third equation, we get $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$. $\therefore$ the eigenvector is $X_2=\begin{pmatrix}1\\1\\1\end{pmatrix}$

Case (iii): When $\lambda=6$, the simultaneous equations $(A-\lambda I)X=0$ becomes

$$\begin{aligned} -3x-y+z&=0 \\ -x-y-z&=0 \\ x-y-3z&=0 \end{aligned}$$

$$\boxed{\begin{array}{ccccccc} & x & & y & & z & \\ -1 & & 1 & & -3 & & -1 \\ -1 & & -1 & & -1 & & -1 \end{array}}$$

Solving first and second equation, we get $\frac{x}{2}=\frac{y}{-4}=\frac{z}{2}$ i.e. $\frac{x}{1}=\frac{y}{-2}=\frac{z}{1}$

$\therefore$ the eigenvector is $X_3=\begin{pmatrix}1\\-2\\1\end{pmatrix}$

Now X_1, X_2, X_3 are pairwise orthogonal.

Consider the modal matrix $M=\begin{pmatrix}-1 & 1 & 1\\ 0 & 1 & -2\\ 1 & 1 & 1\end{pmatrix}$

Then the normalized modal matrix $N=\begin{pmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{pmatrix}$

The required orthogonal transformation that diagonalises A is $N^T AN=D$

$$A\times N=\begin{pmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{pmatrix}\times\begin{pmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{pmatrix}=\begin{pmatrix}-\frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}}\\ 0 & \frac{3}{\sqrt{3}} & -\frac{12}{\sqrt{6}}\\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}}\end{pmatrix}$$

$$N^T \times A \times N = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{3}} & -\frac{12}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$= D(2,3,6)$$

We know that $A^2 = N D^2 N^T$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 36 \end{pmatrix} \times \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -9 & 7 \\ -9 & 27 & 9 \\ 7 & -9 & 11 \end{pmatrix}$$

3. Diagonalise the matrix $A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$ **by orthogonal reduction.**

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of diagonal values of $A\ = 2+6+2 = 10$

$S_2 =$ sum of minors of leading diagonal of A

$$= \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix} = 12 - 12 + 12 = 12$$

$S_3 = |A| = 2(12) + 4(-24) = 24 - 96 = -72$

$\therefore\ \lambda^3 - 10\lambda^2 + 12\lambda + 72 = 0$

By inspection $\lambda = -2$ is a root.

By synthetic division, we have

-2	1	-10	12	72
	0	-2	24	-72
	1	-12	36	0

$\lambda^2 - 12\lambda + 36 = 0$

$(\lambda - 6)^2 = 0$

$\lambda = 6, 6$

$\therefore$ The eigenvalues of A are $\lambda = -2, 6, 6$

Consider the equation $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(2-\lambda)x + 4z = 0$$
$$(6-\lambda)y = 0$$
$$4x + (2-\lambda)z = 0$$

Case (i): When $\lambda = -2$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$4x + 4z = 0$$
$$8y = 0$$
$$4x + 4z = 0$$

	x		y		z	
0		4		4		0
8		0		0		8

Solving first and second equation, we get $\frac{x}{-32} = \frac{y}{0} = \frac{z}{32}$ i.e. $\frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$

$\therefore$ the eigenvector is $X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Case (ii): When $\lambda = 6$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$-4x + 4z = 0$$
$$(0)y = 0$$
$$4x - 4z = 0$$

i.e. we have only one equation $x - z = 0$. Solving we get $X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Case (iii): Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the thirdeigen vector corresponding to $\lambda = 6$ such that

$X_2.X_3^T = 0$ and $X_1.X_3^T = 0$

$\therefore\ a - c = 0$ and $a + c = 0$

	x		y		z	
0		−1		1		0
0		1		1		0

Solving the equations, we get $\frac{a}{0} = \frac{b}{-2} = \frac{c}{0}$ i.e. $\frac{a}{0} = \frac{b}{1} = \frac{c}{0}$

$\therefore$ the third eigen vector is $X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Now X_1, X_2, X_3 are pairwise orthogonal.

Consider the modal matrix $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

Then the normalized modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

The required orthogonal transformation that diagonalises A is $N^T AN = D$

$$A \times N = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{pmatrix}$$

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$$N^T \times A \times N = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D(-2,6,6)$$

3. Find the matrix A whose eigenvalues and eigenvectors are –2, 6, 6 and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ respectively.

We know that $N^T AN = D$ where N is normalized modal matrix, D is a diagonal matrix of eigenvalues.

Modal matrix M $= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ and hence

the normalised modal matrix N $= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

Since N is orthogonal, $N^T = N^{-1}$

Therefore, $N^{-1}AN = D$

$NN^{-1}ANN^{-1} = NDN^{-1}$, pre multiply by N and post multiply by N^{-1}

$A = NDN^{-1}$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \times \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

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$$= \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$$

Exercise

1. Diagonalise the following matrices by means of an orthogonal transformation

(i) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ (iv) $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

2. Diagonalise $A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$ by orthogonal transformation and hence find A^2.

3. Diagonalise the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ if possible.

3. If 1, 3, -4 are the eigen values and $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ are the respective eigen vectors, find the corresponding matrix A.

4. Find the matrix A whose eigen values and eigen vectors are $-1, 1, 4$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ respectively.

Linear Transformation of a Quadratic Form

A homogeneous polynomial in any number of variable is called a Form. A homogeneous polynomial of degree two in any number of variable is called a Quadratic Form.

Example: $x^2 - 2xy + 3y^2$(in 2 variables), $x^2 + 2y^2 - z^2 + 2xy - 3yz + xz$(in 3 variables)

Any quadratic form (say in 3 variables x, y, z) can be expressed as $X^T AX$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and

A= Coefficient of $\begin{pmatrix} x^2 & \frac{xy}{2} & \frac{xz}{2} \\ \frac{xy}{2} & y^2 & \frac{yz}{2} \\ \frac{xz}{2} & \frac{yz}{2} & z^2 \end{pmatrix}$

Example: Write down the matrix of the quadratic form: $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 + 6x_2x_3 - 6x_3x_1$

The matrix of the quadratic form is $A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{pmatrix}$

Let $X^T AX$ be a quadratic form and consider a non singular transformation $X = PY$, P is orthogonal matrix and $Y = \begin{pmatrix} y_1 \\ y_2 \\ : \\ : \\ y_n \end{pmatrix}$.

There fore $\quad X^T AX = (PY)^T A(PY)$

$$= Y^T (P^T AP)Y$$

$$= Y^T (B)Y \qquad \text{where } B = P^T AP$$

Clearly B is a symmetric matrix..

$\therefore$ $Y^T (B)Y$ is also a quadratic form.

i.e. $Y^T (B)Y$ is the linear transformation of the quadratic form under $X = PY$.

If by any non singular linear transformation, a quadratic form be expressed as a sum of squares of the new variables, then the later expression is called the canonical form of the given quadratic form.

Orthogonal reduction of a quadratic form to canonical form

The method of reducing a quadratic form to canonical form is known as orthogonal reduction. Let $X^T AX$ be a quadratic form. Let M be the modal matrix and let N be normalized modal matrix. Since $N = N^T$ and $| N |= 1$, N is an orthogonal matrix. Then the orthogonal transformation $X = NY$ will reduce the quadratic form to $Y^T D(\lambda_1, \lambda_2, ...\lambda_n)Y$ where $\lambda_1, \lambda_2, ...\lambda_n$ are the eigenvalues of A. This method is applicable only if the eigenvectors are linearly independent and are pairwise orthogonal.

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Nature of the Quadratic Form.

Let $X^T AX$ be the given quadratic form and let $Y^T DY$ be its canonical form. Now $Y^T DY$ will contain only r terms, if the rank of A is r. The number of positive terms in the canonical form is called the index and is denoted by s. $\therefore$ the number of non positive terms is $r-s$. The difference between number of positive terms and the non positive terms is called the signature of the quadratic form. i.e. signature = $s-(r-s)=2s-r$. The quadratic form $X^T AX$ in n variables is said to be

i. Positive definite if $r=n,\ s=n.$

ii. Positive semi definite if $r<n,\ s=r$

iii. Negative definite if $r=n,\ s=0$

iv. Negative semi definite if $r<n,\ s=0$

v. Indefinite in all other cases.

Another Method

i. Positive definite if all the eigenvalues of A are positive
ii. Positive semi definite if all the eigenvalues of $A \geq 0$ and at least one eigen value is zero
iii. Negative definite if all the eigenvalues of A are negative
iv. Negative semi definite if all the eigenvalues of $A \leq 0$ and at least one eigenvalue is zero
v. Indefinite if A has positive as well as negative eigenvalues.

Another Method

Let $D_1, D_2, \ldots\ldots, D_n$ are called the principal sub determinants of A. The quadratic form is said to be

i. Positive definite if all $D_1, D_2, \ldots\ldots, D_n \geq 0$

ii. Positive semi definite if some $D_i = 0$ in (i)

iii. Negative definite if $D_1,\ D_3,\ D_5,\ \ldots\ldots\ldots < 0$ and $D_2,\ D_4, D_6, \ldots\ldots\ldots\ldots > 0$

iv. Negative semi definite if if some $D_i = 0$ in (iii)

v. Indefinite in all other cases.

Solved Problems

1. **Reduce the quadratic form $3x^2+3y^2+3z^2+2xy-2yz+2xz$ to canonical form by orthogonal reduction. Also find its rank, signature, index and nature of the quadratic form.**

   Given quadratic form can be expressed as $X^T AX$ .....(1) where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

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and the matrix of the quadratic form is $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

Hint:

Coefficient of $\begin{pmatrix} x^2 & \frac{xy}{2} & \frac{xz}{2} \\ \frac{xy}{2} & y^2 & \frac{yz}{2} \\ \frac{xz}{2} & \frac{yz}{2} & z^2 \end{pmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = sum of diagonal values of $A = 3+3+3 = 9$

S_2 = sum of minors of leading diagonal of A

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$
$$= (9-1) + (9-1) + (9-1)$$
$$= 24$$

$S_3 = |A|$

$$= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$
$$= 3(9-1) - 1(3+1) + 1(-1-3)$$
$$= 24 - 4 - 4$$
$$= 16$$

$\therefore$ the characteristic equation is $\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

By inspection $\lambda = 1$ is a root, since $= 1 - 9 + 24 - 16 = 0$.

By synthetic division, we have

1	**1**	**-9**	**24**	**-16**
	0	**1**	**-8**	**16**
	1	**-8**	**16**	**0**

$$\lambda^2 - 8\lambda + 16 = 0$$
$$(\lambda - 4)(\lambda - 4) = 0$$
$$\lambda = 4, 4$$

$\therefore \lambda = 1, 4, 4$ *are the eigenvalues*

Consider the equations $(A - \lambda I)X = 0$

$$\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$(3-\lambda)x + y + z = 0$$
$$x + (3-\lambda)y - z = 0$$
$$x - y + (3-\lambda)z = 0$$

Case (i) When $\lambda = 1$ the equations $(A - \lambda I)X = 0$ becomes

$$2x + y + z = 0$$
$$x + 2y - z = 0$$
$$x - y + 2z = 0$$

Case (ii) When $\lambda = 4$ the equations $(A - \lambda I)X = 0$ becomes

$$-x + y + z = 0$$
$$x - y - z = 0$$
$$x - y - z = 0$$

Solving the first and second equations, we have

$$\frac{x}{-1-2}=\frac{y}{1+2}=\frac{z}{4-1}$$

$$\frac{x}{-3}=\frac{y}{3}=\frac{z}{3}$$

$\therefore\ X_1=\begin{pmatrix}-1\\1\\1\end{pmatrix}$ is an eigenvector.

Here all the equations are identical. Hence put $x=0,\quad y=1,$ we have $z=-1$

$\therefore\ X_2=\begin{pmatrix}0\\1\\-1\end{pmatrix}$ is the eigen vector.

Case (iii) Let $X_3=\begin{pmatrix}a\\b\\c\end{pmatrix}$ be the third eigenvector.

Here X_2 *and* X_3 are orthogonal. Hence $X_2\ \times\ X_3^T=0$. *i.e.* $0a+b-c=0$

Also X_3 satisfies $x-y-z=0$ and hence $a-b-c=0$

Solving these two equations, we have

$$\frac{a}{-1-1}=\frac{b}{-1-0}=\frac{c}{0-1}$$

$$\frac{a}{-2}=\frac{b}{-1}=\frac{c}{-1}$$

$\therefore\ X_3=\begin{pmatrix}2\\1\\1\end{pmatrix}$ is the eigen vector.

Now X_1, X_2, X_3 are pairwise orthogonal.

Consider the modal matrix $M=\begin{pmatrix}-1&0&2\\1&1&1\\1&-1&1\end{pmatrix}$

Then the normalized modal matrix $N=\begin{pmatrix}\frac{-1}{\sqrt{3}}&0&\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}}&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{6}}\end{pmatrix}$

Now N is orthogonal matrix and hence $N^{-1}=N^T$

Consider the orthogonal transformation $X=NY$ .....(2) where $Y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

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Consider $A \times N = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{8}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-4}{\sqrt{2}} & \frac{4}{\sqrt{6}} \end{pmatrix}$

$$N^T \times A \times N = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{8}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-4}{\sqrt{2}} & \frac{4}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Consider the orthogonal transformation $X = NY$ .....(2) where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Substitute (2) in (1) , we get

$$\begin{aligned} X^T AX &= (NY)^T A(NY) \\ &= Y^T (N^T AN) Y \\ &= Y^T (D) Y \qquad \text{\{by diagonalisation\}} \\ &= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= y_1^2 + 4y_2^2 + 4y_3^2 \end{aligned}$$

Number of terms in the quadratic form is rank, $r = 3$
Number of positive terms in the quadratic form is index, $p = 3$
Signature of the quadratic form is $2p - r = 6 - 3 = 3$. Number of variable $n = 3$
Here $r = n$ and $p = n$. Hence the quadratic form is said to be positive definite.

2. Obtain an orthogonal transformation which will transform the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$ to canonical form.

Given quadratic form can be expressed as $X^T AX$ .....(1) where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

and the matrix of the quadratic form is $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$
i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

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$S_1 =$ sum of diagonal values of $A = 0$ $S_2 =$ sum of minors of leading diagonal of A

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1-1-1 = -3$$

$S_3 = |A| = -1(0-1) + 1(1-0) = 1+1 = 2$

$\therefore\ \lambda^3 - 3\lambda - 2 = 0$

By inspection $\lambda = 2$ is a root.

By synthetic division, we have

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)^2 = 0$

$\lambda = -1, -1$

$$\begin{array}{c|cccc} 2 & 1 & 0 & -3 & -2 \\ & 0 & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$\therefore$ The eigenvalues of A are $\lambda = 2, -1, -1$

Consider the equation $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(0-\lambda)x + y + z = 0$$
$$x + (0-\lambda)y + z = 0$$
$$x + y + (0-\lambda)z = 0$$

Case (i): When $\lambda = 2$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$-2x + y + z = 0$$
$$x - 2y + z = 0$$
$$x + y - 2z = 0$$

$$\begin{array}{cccc} & x & y & z \\ 1 & 1 & -2 & 1 \\ -2 & 1 & 1 & -2 \end{array}$$

Solving first and second equation, we get $\frac{x}{3} = \frac{y}{3} = \frac{z}{3}$ i.e. $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$

$\therefore$ the eigenvector is $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Case (ii): When $\lambda = -1$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$x + y + z = 0$$
$$x + y + z = 0$$
$$x + y + z = 0$$

i.e. we have only one equation $x + y + z = 0$.

Taking $x = 1, z = 0$ we get $y = -1$. Hence $X_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Case (iii): Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the third eigen vector corresponding to $\lambda = -1$ such that

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$X_2.X_3^T = 0$ and $X_1.X_3^T = 0$

$\therefore\ a-b=0$ and $a+b+c=0$

x	y	z	
− 1	0	1	−1
1	1	1	1

Solving, we get $\frac{a}{-1}=\frac{b}{-1}=\frac{c}{2}$ i.e. $\frac{a}{1}=\frac{b}{1}=\frac{c}{-2}$ and hence $X_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

Now X_1, X_2, X_3 are pairwise orthogonal.

Consider the modal matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$

Then the normalized modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$

Now N is orthogonal matrix and hence $N^{-1} = N^T$

Consider the orthogonal transformation $X = NY$ .....(2) where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Consider $A \times N = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$

$$N^T \times A \times N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Substitute (2) in (1) , we get

$$\begin{aligned} X^T AX &= (NY)^T A(NY) \\ &= Y^T (N^T AN)Y \\ &= Y^T (D)Y \end{aligned}$$ {by diagonalisation}

$$= (y_1\ y_2\ y_3)\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}$$

$$= 2y_1^2 - y_2^2 - y_3^2$$

Hence the transformation which transform the given QF to CF is $X = NY$.

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\end{pmatrix} \times \begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}$$

$$x_1 = \frac{1}{\sqrt{3}}y_1 + \frac{1}{\sqrt{2}}y_2 + \frac{1}{\sqrt{6}}y_3$$

$$x_2 = \frac{1}{\sqrt{3}}y_1 - \frac{1}{\sqrt{2}}y_2 + \frac{1}{\sqrt{6}}y_3$$

$$x_3 = \frac{1}{\sqrt{3}}y_1 + 0y_2 - \frac{2}{\sqrt{6}}y_3$$

3. **Reduce the quadratic form** $10x^2 + 2y^2 + 5z^2 + 6yz - 10xz - 4xy$ **to canonical form by orthogonal reduction. Also find its nature.**

Given quadratic form can be expressed as $X^T AX$ .....(1) where $X = \begin{pmatrix}x\\ y\\ z\end{pmatrix}$

and the matrix of the quadratic form is $A = \begin{pmatrix}10 & -2 & -5\\ -2 & 2 & 3\\ -5 & 3 & 5\end{pmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of diagonal values of A $= 10 + 2 + 5 = 17$

$S_2 =$ sum of minors of leading diagonal of A

$$= \begin{vmatrix}10 & -2\\ -2 & 2\end{vmatrix} + \begin{vmatrix}10 & -5\\ -5 & 5\end{vmatrix} + \begin{vmatrix}2 & 3\\ 3 & 5\end{vmatrix} = 16 + 25 + 1 = 42$$

$$S_3 = |A| = 10(10-9) + 2(-10+15) - 5(-6+10) = 10 + 10 - 20 = 0$$

$$\therefore\ \lambda^3 - 17\lambda^2 + 42\lambda = 0$$

$$\lambda(\lambda^2 - 17\lambda + 42) = 0$$

$$\lambda(\lambda - 3)(\lambda - 14) = 0$$

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$\therefore$ The eigenvalues of A are $\lambda = 0, 3, 14$

Consider the equation $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(10-\lambda)x - 2y - 5z = 0$$
$$-2x + (2-\lambda)y + 3z = 0$$
$$-5x + 3y + (5-\lambda)z = 0$$

Case (i): When $\lambda = 0$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$10x - 2y - 5z = 0$$
$$-2x + 2y + 3z = 0$$
$$-5x + 3y + 5z = 0$$

	x		y		z	
− 2		−5		10		−2
2		3		−2		2

Solving first and second equation, we get $\frac{x}{4} = \frac{y}{-20} = \frac{z}{16}$ i.e. $\frac{x}{1} = \frac{y}{-5} = \frac{z}{4}$

$\therefore$ the eigen vector is $X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$

Case (ii): When $\lambda = 3$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$7x - 2y - 5z = 0$$
$$-2x - y + 3z = 0$$
$$-5x + 3y + 2z = 0$$

	x		y		z	
− 2		−5		7		−2
−1		3		−2		−1

Solving first and second equation, we get $\frac{x}{-11} = \frac{y}{-11} = \frac{z}{-11}$ i.e. $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$

Hence $X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Case (iii): When $\lambda = 14$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

$$-4x - 2y - 5z = 0$$
$$-2x - 12y + 3z = 0$$
$$-5x + 3y - 9z = 0$$

	x		y		z	
− 2		−5		−4		−2
−12		3		−2		−12

Solving first and second equation, we get $\frac{x}{-66} = \frac{y}{22} = \frac{z}{44}$ i.e. $\frac{x}{-3} = \frac{y}{1} = \frac{z}{2}$

Hence $X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

Now X_1, X_2, X_3 are pairwise orthogonal.

Consider the modal matrix $M = \begin{pmatrix} 1 & 1 & -3 \\ -5 & 1 & 1 \\ 4 & 1 & 2 \end{pmatrix}$

Then the normalized modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$

Now N is orthogonal matrix and hence $N^{-1} = N^T$

Consider the orthogonal transformation $X = NY$ .....(2) where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Now $A \times N = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{\sqrt{3}} & -\frac{42}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{14}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{28}{\sqrt{14}} \end{pmatrix}$

$$N^T \times A \times N = \begin{pmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix} \times \begin{pmatrix} 0 & \frac{3}{\sqrt{3}} & -\frac{42}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{14}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{28}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

Substitute (2) in (1) , we get

$$X^T AX = (NY)^T A(NY)$$

$$= Y^T (N^T AN) Y$$

$$= Y^T (D) Y \qquad \text{\{by diagonalisation\}}$$

$$= (y_1 \; y_2 \; y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

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$$= 0y_1^2 + 3y_2^2 + 14y_3^2$$

Since the sign of the eigenvalues are positive and zero, the given quadratic form is positive semi definite in nature.

4. Reduce the quadratic form $x^2 + 2y^2 + z^2 - 2yx + 2zy$ to canonical form by orthogonal transformation. Give a non zero set of values x, y, z which makes this quadratic form is zero.

Given quadratic form can be expressed as $X^T AX$ .....(1) where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

and the matrix of the quadratic form is $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = sum of diagonal values of A $= 2 + 1 + 1 = 4$

S_2 = sum of minors of leading diagonal of A

$$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1 + 1 + 1 = 3$$

$S_3 = |A| = 1(2-1) + 1(-1-0) = 1 - 1 = 0$

$\therefore\ \lambda^3 - 4\lambda^2 + 3\lambda = 0$

$\lambda(\lambda - 1)(\lambda - 3) = 0$

$\therefore$ The eigenvalues of A are $\lambda = 0, 1, 3$

Consider the equation $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$(1-\lambda)x - y = 0$
$-x + (2-\lambda)y + z = 0$
$y + (1-\lambda)z = 0$

Case (i): When $\lambda = 0$, the simultaneous equations $(A - \lambda I)X = 0$ becomes

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$$x-y=0$$
$$-x+2y+z=0$$
$$y+z=0$$

$$\begin{array}{ccccccc} & x & & y & & z & \\ -1 & & 0 & & 1 & & -1 \\ 1 & & 1 & & 0 & & 1 \end{array}$$

Solving first and third equation, we get $\frac{x}{-1}=\frac{y}{-1}=\frac{z}{1}$ i.e. $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$

$\therefore$ the eigen vector is $X_1=\begin{pmatrix}1\\1\\-1\end{pmatrix}$

Case (ii): When $\lambda=1$, the simultaneous equations $(A-\lambda I)X=0$ becomes

$$-y=0$$
$$-x+y+z=0$$
$$y=0$$

$$\begin{array}{ccccccc} & x & & y & & z & \\ 1 & & 1 & & -1 & & 1 \\ 1 & & 0 & & 0 & & 1 \end{array}$$

Solving second and third equations, we get $\frac{x}{-1}=\frac{y}{0}=\frac{z}{-1}$ i.e. $\frac{x}{1}=\frac{y}{0}=\frac{z}{1}$

Hence $X_2=\begin{pmatrix}1\\0\\1\end{pmatrix}$

Case (iii): When $\lambda=3$, the simultaneous equations $(A-\lambda I)X=0$ becomes

$$-2x-y=0$$
$$-x-y+z=0$$
$$y-2z=0$$

$$\begin{array}{ccccccc} & x & & y & & z & \\ -1 & & 1 & & -1 & & -1 \\ 1 & & -2 & & 0 & & 1 \end{array}$$

Solving second and third equations, we get $\frac{x}{1}=\frac{y}{-2}=\frac{z}{-1}$ i.e. $\frac{x}{-1}=\frac{y}{2}=\frac{z}{1}$

Hence $X_3=\begin{pmatrix}-1\\2\\1\end{pmatrix}$

Now X_1, X_2, X_3 are pairwise orthogonal.

Consider the modal matrix $M=\begin{pmatrix}1&1&-1\\1&0&2\\1&1&1\end{pmatrix}$

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Then the normalized modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

Now N is orthogonal matrix and hence $N^{-1} = N^T$

Consider the orthogonal transformation $X = NY$ .....(2) where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$\text{Now } A \times N = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{6}} \end{pmatrix}$$

$$N^T \times A \times N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Substitute (2) in (1) , we get

$$X^T AX = (NY)^T A(NY)$$

$$= Y^T (N^T AN) Y$$

$$= Y^T (D) Y \qquad \text{\{by diagonalisation\}}$$

$$= (y_1 \ y_2 \ y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0y_1^2 + 1y_2^2 + 3y_3^2$$

The canonical form of quadratic form is zero when $y_2 = y_3 = 0$ and y_1 is arbitrary.

The orthogonal transformation $X = NY$ becomes

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$$x = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{6}} y_3$$

$$y = \frac{1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_3$$

$$z = -\frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3$$

Taking $y_1 = \sqrt{3},\ y_2 = 0,\ y_3 = 0$, we get $x = 1,\ y = 1,\ z = -1$.

These values makes the quadratic form zero.

5. Determine the nature of the quadratic form $x^2 + 3y^2 + 6z^2 + 2xy + 2yz + 4xz$ without reducing into canonical form.

Let the matrix of the quadratic form be A = $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix}$

Now, $D_1 = |\,1\,| = 1 > 0$

$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$

$= 3 - 1$

$= 2 > 0$

$D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{vmatrix}$

$= 1(18 - 1) - 1(6 - 2) + 2(1 - 6)$

$= 17 - 4 + 2(-5)$

$= 13 - 10 = 3 > 0$

Since all D_1, D_2, D_3 are positive, the given quadratic form is positive definite.

6. Find the nature of the conic $8x^2 - 4xy + 5y^2 = 36$ by reducing the quadratic form $8x^2 - 4xy + 5y^2$ to the form $AX^2 + BY^2$.

The matrix of the quadratic form is $A = \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)(5-\lambda) - 4 = 0$$

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$$\lambda^3 - 13\lambda^2 + 36 = 0$$

$$(\lambda - 4)(\lambda - 9) = 0$$

$\therefore$ The eigen values of A are $\lambda = 4, 9$

$\therefore$ the canonical form is of the form $4X^2 + 9Y^2$

$\therefore$ given conic becomes $4X^2 + 9Y^2 = 36$ i.e. $\frac{X^2}{9} + \frac{Y^2}{4} = 1$ which is an ellipse.

7. State the nature of the quadratic form $2xy + 2yz + 2xz$

The matrix of the quadratic form is $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$D_1 = |0| = 0$$

$$D_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1(0-1) + 1(1-0) = 1 + 1 = 2$$

Here $D_1 = 0$, $D_2 < 0$, $D_3 > 0$. $\therefore$ the quadratic form is indefinite in nature.

Exercise

1. Reduce the quadratic form $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ to canonical form by orthogonal reduction. Also find its nature.

2. Reduce the quadratic form $2x^2 + y^2 + z^2 - 4yz - 2xz + 2xy$ to canonical form by orthogonal reduction. Also find its nature.

3. Determine the nature of the quadratic form $3x^2 - 3y^2 - 5z^2 - 2xy - 6yz - 6xz$ without reducing into canonical form.

4. Reduce the quadratic form $2x^2+5y^2+3z^2+4xy$ to canonical form by an orthogonal transformation. Also find the rank, index and signature of the quadratic form.

5. Reduce the quadratic form $x^2 + 5y^2 + z^2 + 2xy + 2yz + 6xz$ to its canonical form by orthogonal reduction. Also find the signature, index and nature of the quadratic form.

Stretching of Elastic Membrane

In matrix notation, we know that the transformation $Y = AX$ transforms the input X to the output form Y where A is the matrix of the transformation. Consider the following example:

An elastic membrane in the x_1x_2 plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point $P(x_1, x_2)$ goes over into the point $Q(y_1, y_2)$ given by $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = AX = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

In components , the transformation $Y = AX$ becomes

$$y_1 = 5x_1 + 3x_2$$
$$y_2 = 3x_1 + 5x_2$$

From the transformation, $X = A^{-1}Y$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{|A|} adj(A) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$x_1 = \frac{1}{16}(5y_1 - 3y_2)$$

$$x_2 = \frac{1}{16}(-3y_1 + 5y_2)$$

Substitute x_1, x_2 in the given circle $x_1^2 + x_2^2 = 1$ to get the transformed conic in y_1, y_2

$$\frac{1}{16^2}(5y_1 - 3y_2)^2 + \frac{1}{16^2}(-3y_1 + 5y_2)^2 = 1$$

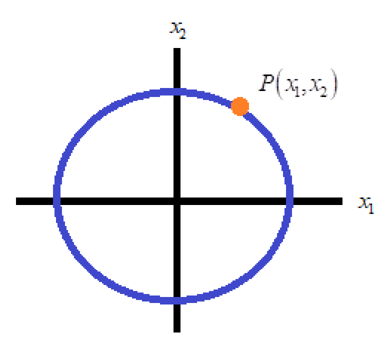
$$25y_1^2 + 9y_2^2 - 30y_1y_2 + 9y_1^2 + 25y_2^2 - 30y_1y_2 = 256$$

$$34y_1^2 + 34y_2^2 - 60y_1y_2 = 256$$

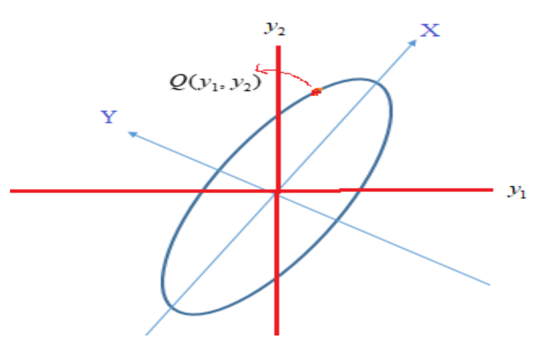
which is the transformed conic.

Removing the y_1y_2 term with the usual method, the conic becomes $\frac{X^2}{16} + \frac{Y^2}{1} = 4,$ which is an ellipse.

Thus a circle is deformed to an ellipse by the transformation.



Given Shape



Transformed Shape

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Application of Eigen Value, Eigen Vector in Stretching of Elastic Membrane

The transformation $Y = AX$ deforms the shape of a conic. But we know that $AX = \lambda X$. This gives the eigenvalues and eigenvectors of A. Hence it can be applied to the study of stretching of a circular membrane.

Consider the above example once again:

An elastic membrane in the x_1x_2 plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point $P(x_1, x_2)$ goes over into the point $Q(y_1, y_2)$ given by $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = AX = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Find the principal directions of the position vector X of P for which the direction of the position vector Y of Q. Also find the shape of the circle after stretching.

We have to find vectors X such that $Y = \lambda X$. Since $Y = AX$, we have $AX = \lambda X$, which is the equation of getting eigen vector corresponding to the eigen value λ. Hence we shall find the eigenvalues first by using $|A - \lambda I| = 0$.

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$
$$25 - 10\lambda + \lambda^2 - 9 = 0$$
$$\lambda^2 - 10\lambda + 16 = 0$$
$$(\lambda - 2)(\lambda - 8) = 0$$
$$\lambda = 2, 8$$

Hence the eigenvalues are $\lambda = 2, 8$. Now consider the system of equations $(A - \lambda I)X = 0$

$$(5-\lambda)x_1 + 3x_2 = 0$$
$$3x_1 + (5-\lambda)x_2 = 0$$

$$If \;\; \lambda = 2, \quad 3x_1 + 3x_2 = 0 \;\;\&\;\; 3x_1 + 3x_2 = 0$$
$$i.e. \;\; x_1 = -x_2$$
$$\therefore X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \; is\, an\ eigenvector$$

$$If \;\; \lambda = 8, \quad -3x_1 + 3x_2 = 0 \;\;\&\;\; 3x_1 - 3x_2 = 0$$
$$i.e. \;\; x_1 = x_2$$
$$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \; is\, an\ eigenvector$$

Since $\theta_1 = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = 135° \;\; and \;\; \theta_2 = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}1 = 45°$

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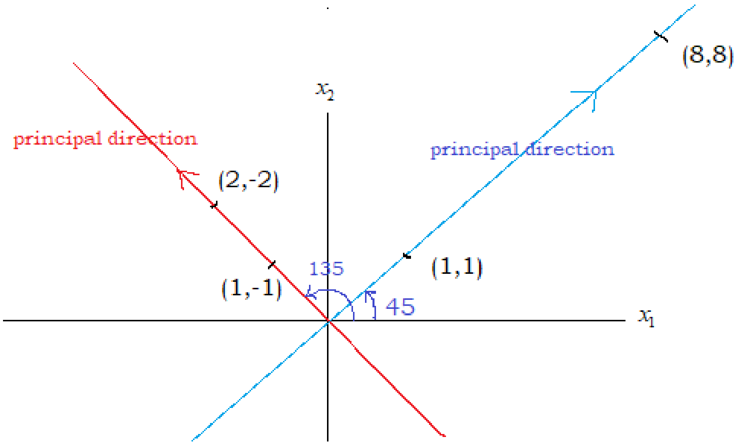
These vectors make 135° and 45° angles with the positive x_1 direction. They give the principal directions. The eigen values show that in the principal directions the membrane is stretched by factors 2 and 8 respectively.



Accordingly, if we choose the principal directions as directions of a new y_1y_2 Coordinate system, say, with the positive y_1 semi axis in the first quadrant and the positive y_2 semi axis in the second quadrant of the x_1x_2 system. Hence, we have, if we set $y_1 = \lambda_1 \cos\phi,\ y_2 = \lambda_2 \sin\phi$, then a boundary point of the unstretched circular membrane has coordinates $\cos\phi,\ \sin\phi$. Hence after the stretch we have

$$y_1 = 2\cos\phi,\ y_2 = 8\sin\phi$$

$$\frac{y_1}{2} = \cos\phi,\ \ \frac{y_2}{8} = \sin\phi$$

$$\frac{y_1^2}{4} + \frac{y_2^2}{64} = \cos^2\phi + \sin^2\phi$$

$$\frac{y_1^2}{4} + \frac{y_2^2}{64} = 1$$

This shows that the deformed shape is an ellipse.

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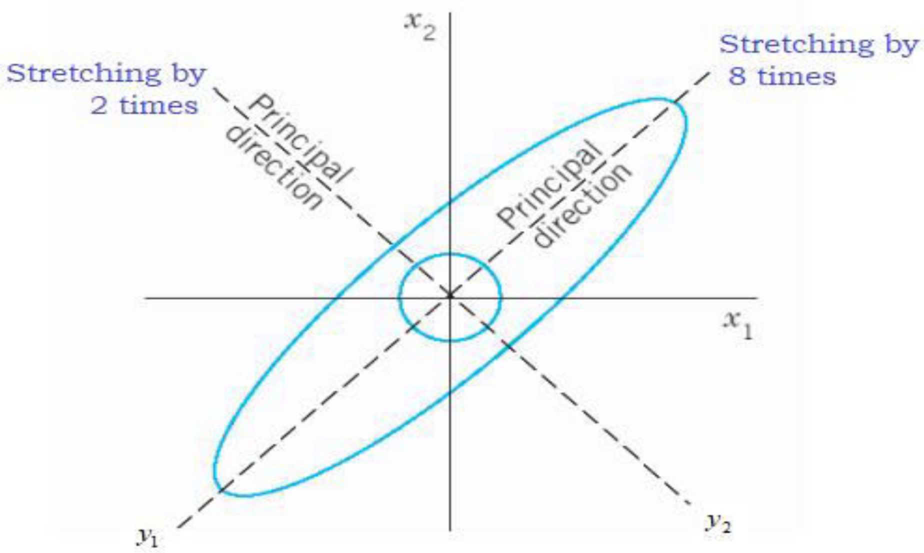

Given conic circle and transformed conic ellipse

Given matrix $A = \begin{pmatrix} 3 & 1.5 \\ 1.5 & 3 \end{pmatrix}$ **in a deformation** $Y = AX$ **where** $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, **find the principal directions of the position vector X of P for which the direction of the position vector Y of Q and the corresponding factors of extension or contractions.**

We have to find vectors X such that $Y = \lambda X$. Since $Y = AX$, we have $AX = \lambda X$, which is the equation of getting eigen vector corresponding to the eigen value λ. Hence we shall find the eigenvalues first by using $|A - \lambda I| = 0$.

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$$\begin{vmatrix} 3-\lambda & \frac{3}{2} \\ \frac{3}{2} & 3-\lambda \end{vmatrix} = 0$$

$$9-6\lambda+\lambda^2-\frac{9}{4}=0$$

$$\lambda^2-6\lambda+\frac{27}{4}=0$$

$$\lambda^2-\frac{3}{2}\lambda-\frac{9}{2}\lambda+\frac{27}{4}=0$$

$$\lambda\left(\lambda-\frac{3}{2}\right)-\frac{9}{2}\left(\lambda-\frac{3}{2}\right)=0$$

$$\left(\lambda-\frac{3}{2}\right)\left(\lambda-\frac{9}{2}\right)=0$$

$$\lambda=\frac{3}{2},\frac{9}{2}$$

Hence the eigenvalues are $\lambda=\frac{3}{2},\frac{9}{2}$. Now consider the system of equations $\left(A-\lambda I\right)X=0$

$$\left(3-\lambda\right)x_1+1.5x_2=0$$
$$1.5x_1+\left(3-\lambda\right)x_2=0$$

When $\lambda=1.5,$ the equations $\left(A-\lambda I\right)X=0$ becomes

$$1.5x_1+1.5x_2=0 \quad \& \quad 1.5x_1+1.5x_2=0$$
$$i.e.\ x_1=-x_2$$
$$\therefore X_1=\begin{pmatrix}-1\\1\end{pmatrix} \text{ is an eigenvector}$$

When $\lambda=4.5,$ the equations $\left(A-\lambda I\right)X=0$ becomes

$$-1.5x_1+3x_2=0 \quad \& \quad 3x_1-1.5x_2=0$$
$$i.e.\ x_1=x_2$$
$$\therefore X_2=\begin{pmatrix}1\\1\end{pmatrix} \text{ is an eigenvector}$$

Since $\theta_1=\tan^{-1}\left(\frac{1}{-1}\right)=\tan^{-1}(-1)=135^\circ$ *and* $\theta_2=\tan^{-1}\left(\frac{1}{1}\right)=\tan^{-1}1=45^\circ$

These vectors make 135° and 45° angles with the positive x_1 direction. They give the principal directions. The eigen values show that in the principal directions the membrane is stretched by

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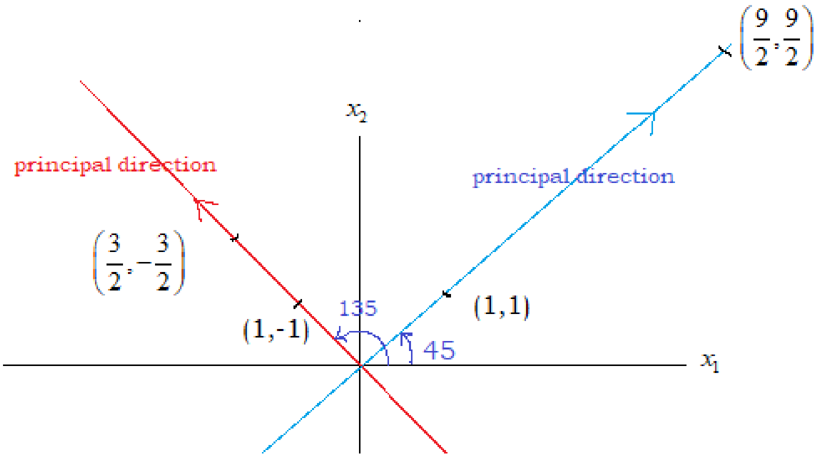
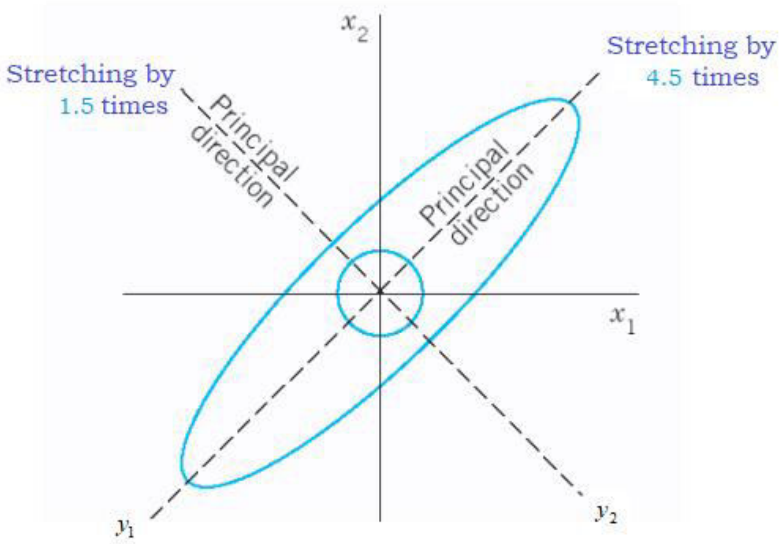
factors $\frac{3}{2}$ and $\frac{9}{2}$ respectively.





Given matrix $A = \begin{pmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}$ **in a deformation** $Y = AX$ **where** $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ *and* $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, **find the principal directions of the position vector X of P for which the direction of the position vector Y of Q and the corresponding factors of extension or contractions.**

We have to find vectors X such that $Y = \lambda X$. Since $Y = AX$, we have $AX = \lambda X$, which is the equation of getting eigen vector corresponding to the eigen value λ. Hence we shall find the eigenvalues first by using $|A - \lambda I| = 0$.

$$\begin{vmatrix} 7-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{vmatrix} = 0$$

$$14-9\lambda+\lambda^2-6=0$$

$$\lambda^2-9\lambda+8=0$$

$$(\lambda-1)(\lambda-8)=0$$

$$\lambda=1,\ 8$$

Hence the eigenvalues are $\lambda = 1, 8$. Now consider the system of equations $(A-\lambda I)X=0$

$$(7-\lambda)x_1+\sqrt{6}\,x_2=0$$

$$\sqrt{6}\,x_1+(2-\lambda)x_2=0$$

When $\lambda=1,$ the equations $(A-\lambda I)X=0$ becomes

$$6x_1+\sqrt{6}\,x_2=0$$

$$\sqrt{6}\,x_1+x_2=0$$

Both represents the same equations $\sqrt{6}\,x_1+x_2=0$ and hence

$$x_2=-\sqrt{6}\,x_1$$

$$when\ x_1=-1,\quad x_2=\sqrt{6}$$

$\therefore X_1=\begin{pmatrix}-1\\ \sqrt{6}\end{pmatrix}$ is an eigen vector.

When $\lambda=8,$ the equations $(A-\lambda I)X=0$ becomes

$$-1x_1+\sqrt{6}\,x_2=0$$

$$\sqrt{6}\,x_1-6x_2=0$$

Both represents the same equations $-1x_1+\sqrt{6}\,x_2=0$ and hence

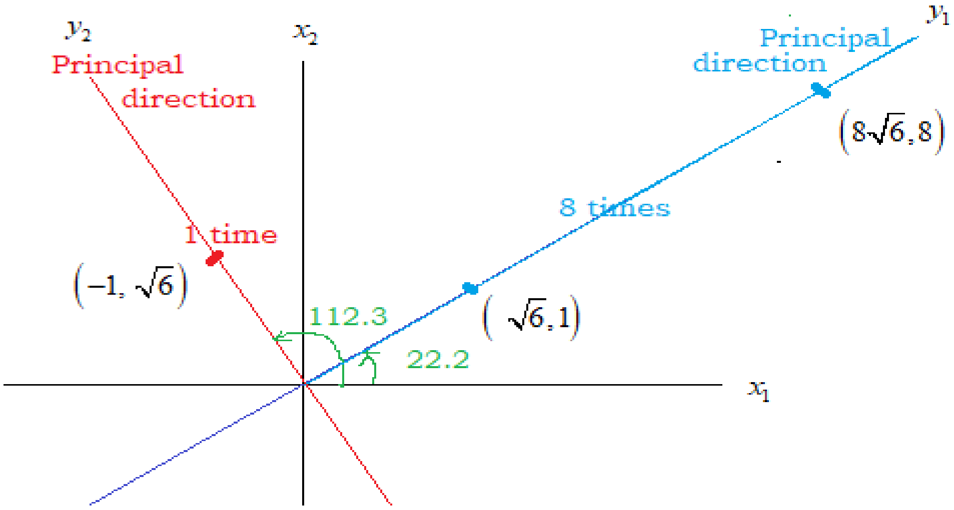
$$x_1=\sqrt{6}\,x_2$$

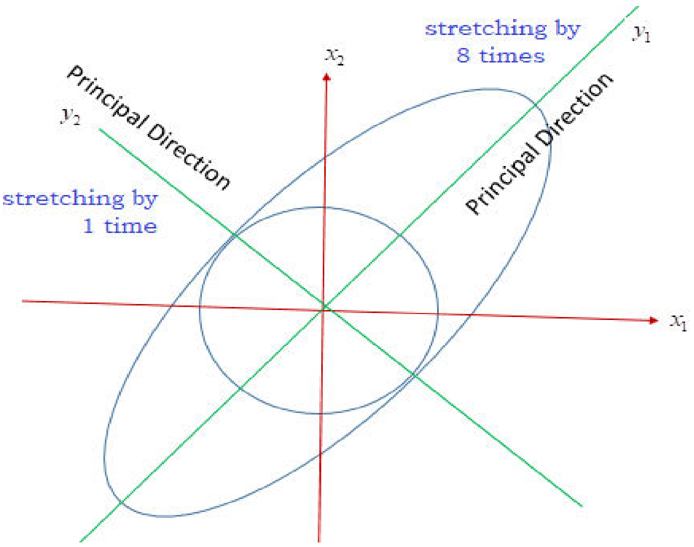
$$when\ x_2=1,\quad x_1=\sqrt{6}$$

$\therefore X_2=\begin{pmatrix}\sqrt{6}\\ 1\end{pmatrix}$ is an eigen vector.

Since $\theta_1=\tan^{-1}\left(-\frac{\sqrt{6}}{1}\right)=\tan^{-1}(-\sqrt{6})=-67.7°=112.3°\ \ and\ \ \theta_2=\tan^{-1}\left(\frac{1}{\sqrt{6}}\right)=22.2°$

These vectors make 112.3° and 22.2° angles with the positive x_1 direction. They give the principal directions. The eigen values show that in the principal directions the membrane is stretched by factors 1 and 8 respectively.





Exercise

Given matrix A in a deformation $Y = AX$, find the principal directions of the position vector X of P for which the direction of the position vector Y of Q and the corresponding factors of extension or contractions.

$(i)\ A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$ $(ii) \begin{pmatrix} 5 & 2 \\ 2 & 13 \end{pmatrix}$ $(iii) \begin{pmatrix} 2 & 0.4 \\ 0.4 & 2 \end{pmatrix}$ $(iv) \begin{pmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{pmatrix}$

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UNIT II - DIFFERENTIAL CALCULUS

Representation of Functions

Quantity: Anything which is capable of being measured or which can be divided into parts (in general operations of Mathematics) is called a quantity.

A **constant** is a quantity which retains the same value throughout a mathematics investigation. A constant may have fixed value, for example, 10, 15, 100, etc or may be supposed to have a fixed value in any particular investigation. E.g. a, b, c in the equation of straight line, ax+by+c=0. The first one is called absolute constants and the second one is arbitrary constants.

A **variable** is a quantity which is capable of assuming different values that may be assigned to it.

Function: Functions provide us a convenient way to handle a relationship between a variable that depends on the value of another variable. If $y = x^3 + 3x$, we see that when $x = 0,\ 1,\ 2, \ldots..$ $y = 0,\ 4,\ 14, \ldots..$ Here x and y are connected that if we make any change in the value of x, there is a change in the value of y. When the quantities are connected in this manner, one is said to be a function of other. In this example y is a function of x. Also x is the independent variable and y is the dependent variable.

Definition: A **function** consists of a domain and a rule. The **domain** is a set of real numbers. The rule assigns to each number in the domain one and only one number.

Functions are normally by $f\ or\ g$ and the elements in the domain are denoted by $x,\ t,\ a.$ The value assigned by a function A function f to a member x of its domain is written as $f(x)$. The collection of $f(x)$ is called the range of f.

Note:
1. A function must make an assignment to each number in the domain
2. A function can assign only one number to any given number in the domain.

Examples of Functions:
Let f be a function whose domain consists of all real numbers and the rule assigns for any real x, the number $x^3 + 1$. Then we write $f(x) = x^3 + 1$, for all x.

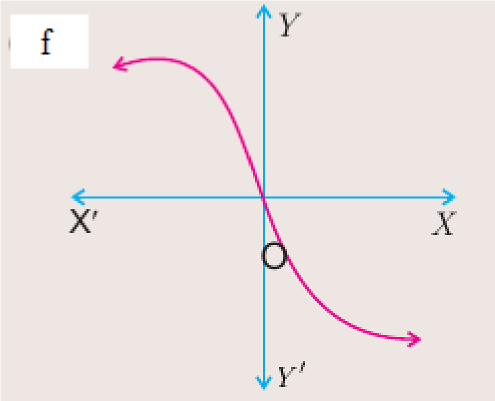
Let Page | **63** f be a function whose domain consists of all real numbers except 1 and the rule assigns for any real $x \neq 1$, the number $\frac{x+1}{x-1}$. Then we write $f(x) = \frac{x+1}{x-1}$, for $x \neq 1$.

Note: A function can be represented by an equation, graph, table or symbolic form in words

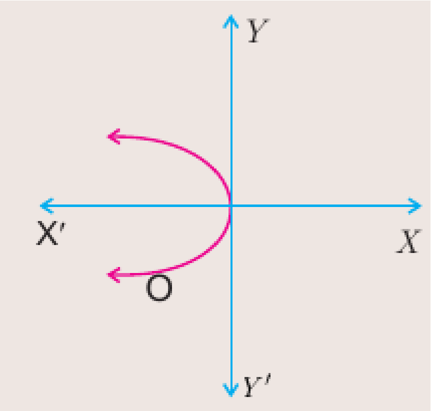
We denote the function of x by the symbols as y, f(x), F(x), etc. For example,
$y = x^2 + 3 \text{ or } f(x) = x^2 + 3, \ f : x \to y$.

Vertical Line Test: A curve in the xy plane is the graph of the function $f(x)$ if and only if no vertical line intersects the curve more than once.

Check whether the following curves represent a function.



By vertical line test, it represent a function.



By vertical line test, it does not represent a function.

Guess: Can a horizontal line pass through more than one point on the graph of a function? Explain.

Example: Describe the function f that associates with each temperature in degrees Celsius the corresponding temperature in degrees Fahrenheit.

Let x be the temperature in Celsius. Then the temperature in Fahrenheit is F(x).

Therefore $F(x) = \frac{9}{5}x + 32$. When Celsius $x = 0$, the corresponding Fahrenheit value is $F = \frac{9}{5}(0) + 32 = 32$

Example: If $f(x) = x^2 - 2x + 3$, find the value of $f(0), f(-3), f(2y)$.

$f(0) = (0)^2 - 2(0) + 3 = 3$

$f(-3) = (-3)^2 - 2(-3) + 3 = 9 + 6 + 3 = 18$

$f(2y) = (2y)^2 - 2(2y) + 3 = 4y^2 - 4y + 3$

Example: A function is defined by $f : x \to ax + b$, where a and b are numbers. If $f(1) = 2$ & $f(2) = -1$, what is $f(3)$?.

Given $f(1) = 2$ & $f(2) = -1$

$a + b = 2$ & $2a + b = -1$

Subtracting we get $a = -3$

Substituting $a = -3$ in $a + b = 2$, we get $b = 5$

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Therefore $f(x)=-3x+5$ & *hence* $f(3)=-3\times 3+5=-4$

Example: If $y=f(x)=\dfrac{3x-3}{4x-3}$**, show that** $x=f(y)$.

Given

$$\begin{aligned}
&y=\frac{3x-3}{4x-3}\\
&y(4x-3)=3x-3\\
&4xy-3y=3x-3\\
&4xy-3x=3y-3\\
&x(4y-3)=3y-3\\
&x=\frac{3y-3}{4y-3}=f(y)
\end{aligned}$$

Example: If $y=f(x)=\dfrac{1+x}{1-x}$ *and* $z=f(y)$, **find** z **as a function of** x.

Given

$$\begin{aligned}
z&=f(y)\\
&=\frac{1+y}{1-y}\\
&=\frac{1+\dfrac{1+x}{1-x}}{1-\dfrac{1+x}{1-x}}\\
&=\frac{\dfrac{1-x+1+x}{1-x}}{\dfrac{1-x-1-x}{1-x}}\\
&=-\frac{1}{2x}
\end{aligned}$$

Example: Find the domain and codomain of the function $f(x)=\sqrt{x-2}$.

Here value of $f(x)$ exist for all values of $x\geq 2$. Hence the domain of the function is $2\leq x\leq\infty$. Also when $x=2$ the value of $f(x)$ is 0 and when $x>2$, the value of $f(x)>0$. Hence the range is $0\leq x\leq\infty$.

Example: Let $f(x)=\dfrac{x^2+x-2}{x^2+5x-6}$. **Find the domain of** f **.**

The denominator x^2+5x-6 can be factored as $(x-1)(x+6)$. Therefore the denominator is 0 for $x=1$ *and* $x=-6$. Thus the domain of f consists all numbers except 1 *and* -6.

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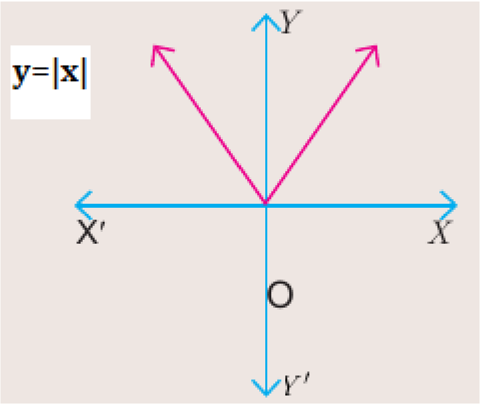
Note: $f(x)=\frac{x^2+x-2}{x^2+5x-6}=\frac{(x-1)(x+2)}{(x-1)(x+6)}=\frac{x+2}{x+6}$. Although the expression $\frac{x+2}{x+6}$ is valid for $x=1$, the number 1 is not in the domain of f.

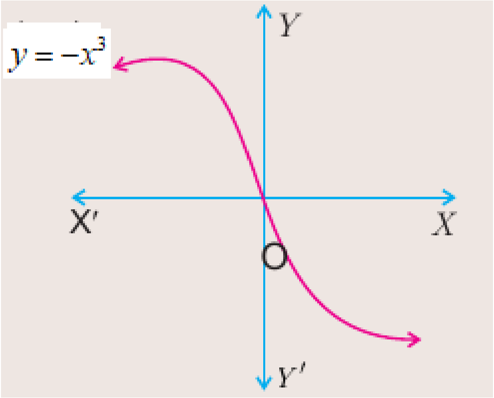
Example: Find the domain of the function $f(x)=\frac{2x^3-5}{x^2+x-6}$.

The denominator x^2+x-6 can be factored as $(x-2)(x+3)$. Therefore the denominator is 0 for $x=2$ *and* $x=-3$. Thus the domain of f consists all numbers except 2 *and* -3.

Even Function: A function $f(x)$ is said to be even if $f(-x)=f(x)$ for all x in its domain.
Note: Even functions are symmetric with x axis.
Example: x^2, $\cos x$, $|x|$ are even functions.



Odd Function: A function $f(x)$ is said to be odd if $f(-x)=-f(x)$ for all x in its domain.
Note: Odd functions are symmetric with origin.
Example: x, x^3, $\sin x$ are odd functions.



Example: Check whether the following functions is odd or even.

$(i)\ x\sin x+x^2$ $\quad (ii)\ \frac{\cos x}{x^3}$ $\quad (ii)\ x^2-2x+1$

$(i)\ f(x)=x\sin x+x^2$

$f(-x)=-x\sin(-x)+(-x)^2$

$=x\sin x+x$

$=f(x)$

$\therefore f(x)\ is\ even$

$(ii)\ f(x)=\frac{\cos x}{x^3}$

$f(-x)=\frac{\cos(-x)}{(-x)^3}$

$=-\frac{\cos x}{x^3}$

$=-f(x)$

$\therefore f(x)\ is\ odd$

$(iii)\quad f(x)=x^2-2x+1$

$f(x)=(-x)^2-2(-x)+1$

$=x^2+2x+1$

$\neq f(x)\ or\ -f(x)$

$\therefore f(x)\ is\ neither\ odd\ nor\ even$

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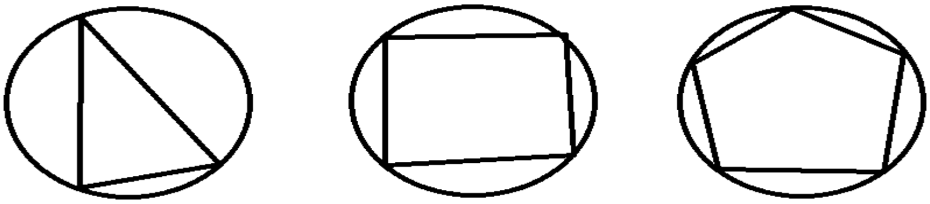
Limits



(a) Let a regular polygon be inscribed in a circle of given radius. Consider the following points.

i. The area of the polygon cannot be greater than the area of circle however large the number of sides may be.

ii. As the number of sides of the polygon increases indefinitely, the area of the polygon continually approaches the area of the circle.

iii. The difference between the area of circle and the area of the polygon can be made as small as we increasing the number of sides of the polygon.

This is expressed in calculus by saying that the limit of the area of polygon inscribed in a circle, as the number of sides increases indefinitely (or approaches infinity), is the area of circle.

(b) Consider the series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots$ which is a G.P. Let S_n be the sum of n terms. Then

$S_n = \dfrac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right) = 2-\frac{1}{2^{n-1}}$. Here we notice that

(i) S_n can never be greater than 2.

(ii) S_n continually approaches 2 as n increases.

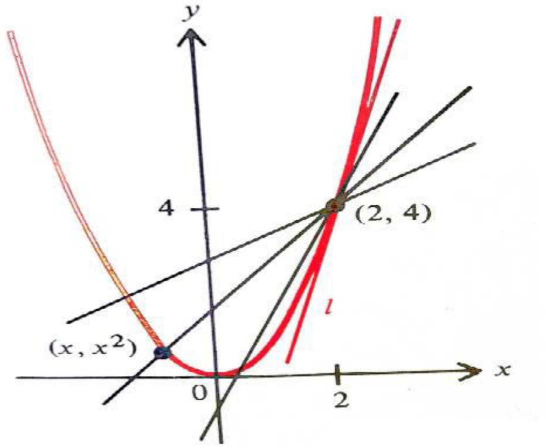
(ii) The difference between 2 and S_n can be made as small as we increases the value of n.

This is expressed in symbols as $\underset{n\to\infty}{Lt} S_n = 2 \quad or \quad S_n \to 2 \ as \ n \to \infty.$

(c) If $f(x)=x^2$, then the slope of the line through the points $(2,4)$ and (x,x^2) on the graph of f is $\frac{x^2-4}{x-2}$.

AS x approaches 2, these lines become closer to the line l which is the tangent line of f at $(2,4)$.

Thus the slope of l is obtained as the limit of the slope $\frac{x^2-4}{x-2}$. I.e. $\underset{x\to 2}{Lt}\frac{x^2-4}{x-2}$



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(d) Let us find the limit of a function $\frac{2}{3+x}$

Case(1): Let x be greater than 0.	**Case(2):** Let x be less than 0.
(i) When x is greater than 0, $\frac{2}{3+x}$ is less than $\frac{2}{3}$ (ii) When x approaches to 0, $\frac{2}{3+x}$ comes nearer to $\frac{2}{3}$ (iii) The difference between $\frac{2}{3+x}$ and $\frac{2}{3}$ can be made less by giving smaller value for x. Therefore we say that the limiting value of $\frac{2}{3+x}$ is $\frac{2}{3}$ when x approaches 0 through real numbers greater than 0. In symbol we write $\underset{x\to 0+}{Lt}\frac{2}{3+x}=\frac{2}{3}$......(1)	(i) When x is less than 0, $\frac{2}{3+x}$ is greater than $\frac{2}{3}$ (ii) When x approaches to 0, $\frac{2}{3+x}$ comes nearer to $\frac{2}{3}$ (iii) The difference between $\frac{2}{3+x}$ and $\frac{2}{3}$ can be made less by giving smaller value for x. Therefore we say that the limiting value of $\frac{2}{3+x}$ is $\frac{2}{3}$ when x approaches 0 through real numbers less than 0. In symbol we write $\underset{x\to 0-}{Lt}\frac{2}{3+x}=\frac{2}{3}$........(2)

Combining (1) and (2), we write $\underset{x\to 0}{Lt}\frac{2}{3+x}=\frac{2}{3}$.

(e) Consider the expression $\frac{\sin x}{x}$

To obtain the limit let us evaluate the values of the function for certain values of x close to 0.

x	$\pm\frac{\pi}{4}$	$\pm\frac{\pi}{40}$	$\pm\frac{\pi}{400}$	$\pm\frac{\pi}{4000}$
$\frac{\sin x}{x}$	0.900316	0.99897	0.9999897	0.99999989

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From the table, we observe that $\frac{\sin x}{x}$ approaches 1 as x approaches 0. Therefore $\underset{x\to 0}{Lt}\frac{\sin x}{x}=1$

Definition: A function $f(x)$ is said to tend to a limit l when x tends to $'a'$ if the difference between $f(x)$ and l can be made as small as we please by making x sufficiently near $'a'$ and we write

$$\underset{x\to a}{Lt}\, f(x)=l.$$

Note:

(i) $\frac{1}{x}\to +\infty$ as $x\to 0+$ (ii) $\frac{1}{x}\to -\infty$ as $x\to 0-$ (iii) $\frac{1}{x}\to 0$ as $x\to \pm\infty$

Find $\underset{x\to 1}{Lt}\frac{x^3-1}{x-1}$

$\underset{x\to 1}{Lt}\frac{x^3-1}{x-1}=\frac{0}{0}$, indeterminant form.

Therefore put $x=1+h$. i.e. $x\to 1$ means $h\to 0$

$$\underset{x\to 1}{Lt}\frac{x^3-1}{x-1}=\underset{h\to 0}{Lt}\frac{(1+h)^3-1}{(1+h)-1}$$

$$=\underset{h\to 0}{Lt}\frac{1+3h+3h^2+h^3-1}{h}$$

$$=\underset{h\to 0}{Lt}3+h+h^2=3$$

Find $\underset{x\to 1}{Lt}\frac{x^3-1}{x-1}$

$$\underset{x\to 1}{Lt}\frac{x^3-1}{x-1}=\underset{x\to 1}{Lt}\frac{(x-1)(x^2+x+1)}{x-1}$$

$$=\underset{x\to 1}{Lt}(x^2+x+1)$$

$$=1+1+1=3$$

Distinction between limit and value

Let us discuss the following cases to understand the difference between the limit and value of the function

(i) Value and limit are different

Consider $\underset{x\to 3}{Lt}\frac{x^2-9}{x-3}=\underset{x\to 3}{Lt}\frac{\left(x-3\right)\left(x+3\right)}{x-3}=\underset{x\to 3}{Lt}(x+3)=3+3=6.$

Value of $\frac{x^2-9}{x-3}$ when $x=3$ is equal to $\frac{0}{0}$ which is indeterminate

(ii) Value and limit are same

$$\underset{x \to 0}{Lt} x^2 + 3x + 5 = 0 + 0 + 5 = 5.$$

Value of $x^2 + 3x + 5$ when $x = 0$ is also 0+0+5=5.

(iii) Both the limit and value of the function exist but may not be equal.

Consider $f(x) = \begin{vmatrix} 0, & if \ \ x \neq 1 \\ 1, & if \ \ x = 1 \end{vmatrix}$

Here the value at 1 is 1. i.e. $f(1) = 1$. But the limit at 1 is 0. i.e. $\underset{x \to 1}{Lt} f(x) = 0$

(The limit of a function at a point does not depend on the value at that point. It depends only on the values taken at nearby points.)

(iv) Sometimes value of the function may exist but limit may not exist

Find $\underset{x \to 3}{Lt} [x].$

Here value of the function $f(3) = [3] = 3$

The numbers 3.01, 3.002, 3.0003 and 2.99, 2.998 are very close to 3. If we take the values of $f(x)$ at these points, they are 3, 3, 3, 2, 2. Not all of them are close to 2 and not all of them are close to 3. Therefore the limit does not exist.

(v) Sometimes limit may exist at a point, but the point may not in the domain of $f(x)$.

Find $\underset{x \to 0}{Lt} x \sin \frac{1}{x}.$

Here the function is not defined at $x = 0.$

We know that $-1 \leq \sin \frac{1}{x} \leq 1$. Therefore $-x \leq x.\sin \frac{1}{x} \leq x$. Taking limit throught the inequality, we have

$$\underset{x \to 0}{Lt} x \sin \frac{1}{x} = 0.$$

Left Limit of $f(x)$ is denoted by $\underset{x \to a^-}{Lt} f(x) = l$ as x approaches a from the left.	**Right Limit** of $f(x)$ is denoted by $\underset{x \to a^+}{Lt} f(x) = l$ as x approaches a from the right.
Thus the limit of $f(x)$ exists if left and right limits are equal.	

Consider the Heaviside function	When t approaches 0 from the left, $H(t)$

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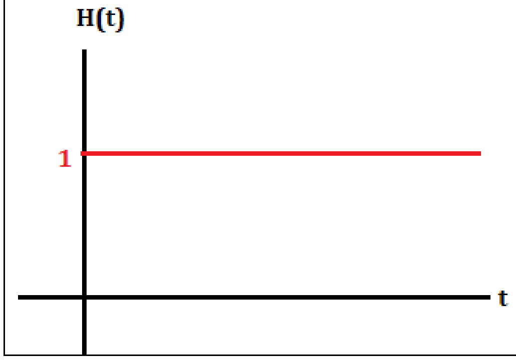
$H(t)=\begin{vmatrix}0, & if \;\; t<0 \\ 1, & if \;\; t\geq 0\end{vmatrix}$  	approaches 0. When t approaches 0 from the right, $H(t)$ approaches 1. $\therefore \underset{t\to 0}{Lt} H(t)$ does not exist.

Note: The line $x=a$ is called the vertical asymptote of the curve $f(x)$ if its limit approaches ∞ if $x\to a$.

Find the value of $\underset{x\to 3^-}{Lt}\frac{2x}{x-3}$ When x approaches 3 from left (i.e. smaller than 3), denominator becomes very small negative number but numerator is 6 and hence $\underset{x\to 3^-}{Lt}\frac{2x}{x-3}=-\infty$	**Find the value of** $\underset{x\to 3^+}{Lt}\frac{2x}{x-3}$ When x approaches 3 from right (i.e. larger than 3), denominator becomes very small positive number but numerator is 6 and hence $\underset{x\to 3^+}{Lt}\frac{2x}{x-3}=\infty$

.

Determine the infinite limit of $\underset{x\to -3^-}{Lt}\frac{2+x}{x+3}$ When x approaches -3 from left (i.e. smaller than -3), denominator becomes very small negative number but numerator is -1 and hence $\underset{x\to -3^-}{Lt}\frac{2+x}{x+3}=\infty$	**Determine the infinite limit of** $\underset{x\to -3^+}{Lt}\frac{2+x}{x+3}$ When x approaches -3 from right (i.e. larger than -3), denominator becomes very small positive number but numerator is -1 and hence $\underset{x\to -3^+}{Lt}\frac{2+x}{x+3}=-\infty$

CALCULATING LIMITS USING THE LIMIT LAWS

In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

I. $\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)]=c\lim_{x\to a} f(x)$

4. $\lim_{x\to a}[f(x)g(x)]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

5. $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ if $\lim\limits_{x\to a} g(x) \neq 0$

If $f(x) = g(x)$ when $x \neq a$, then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$, provided the limits exist.

If $f(x) \le g(x)$ for all x, then $\underset{x\to a}{Lt} f(x) \le \underset{x\to a}{Lt} g(x)$

Rules to find the limit

(i) Take the value of $f(x)$ at $x = a$. If it is not indeterminate, it is the limit at a.

(ii) If it is indeterminate, cancel out the common factor in the numerator and denominator and then take the value at a.

(iii) Or expand the products, simplify and take the value at a.

(iv) Sometimes conjugate surds can be used to simplify and then take the value at a.

(v) Some standard formulas for finding limits can be applied.

Solved Problems on Limit of a Function

1. **Evaluate** (i) $\underset{x\to 0}{Lt}\left(x^2+\cos x\right)$ (ii) $\underset{x\to 0}{Lt}\left(x\cos x\right)$ (iii) $\underset{x\to 0}{Lt}\frac{\left(x\cos x\right)}{\left(x^2+\cos x\right)}$

(i) $\underset{x\to 0}{Lt}\left(x^2+\cos x\right)$

$$= \underset{x\to 0}{Lt}\left(x^2\right) + \underset{x\to 0}{Lt}\left(\cos x\right)$$

$$= 0 + 1 = 1$$

(ii) $\underset{x\to 0}{Lt}\left(x\cos x\right)$

$$= \underset{x\to 0}{Lt}\left(x\right) \times \underset{x\to 0}{Lt}\left(\cos x\right)$$

$$= 0 \times 1 = 0$$

(iii) $\underset{x\to 0}{Lt}\frac{\left(x\cos x\right)}{\left(x^2+\cos x\right)}$

$$= \frac{\underset{x\to 0}{Lt}\left(x\cos x\right)}{\underset{x\to 0}{Lt}\left(x^2+\cos x\right)}$$

$$= \frac{0}{1} = 0$$

.

2. **Evaluate** $\underset{x\to 1}{Lt}\frac{x^2-4x}{x^2-3x-4}$

3. **Evaluate** $\underset{x\to -1}{Lt}\frac{2x^2+3x+1}{x^2-2x-3}$

$$\underset{x\to 1}{Lt}\ \frac{x^2-4x}{x^2-3x-4}=\underset{x\to 1}{Lt}\ \frac{x(x-4)}{(x-4)(x+1)}$$

$$=\underset{x\to 1}{Lt}\ \frac{x}{(x+1)}$$

$$=\frac{1}{2}$$

$$\underset{x\to -1}{Lt}\ \frac{2x^2+3x+1}{x^2-2x-3}=\underset{x\to -1}{Lt}\ \frac{(2x+1)(x+1)}{(x-3)(x+1)}$$

$$=\underset{x\to -1}{Lt}\ \frac{(2x+1)}{(x-3)}$$

$$=\frac{-1}{-4}=\frac{1}{4}$$

.

4 Prove that $\underset{x\to 0}{Lt}\, x^4\cos\frac{2}{x}=0.$

We know that $-1\le\cos\frac{1}{x}\le 1.$

Therefore $-1\le\cos\frac{2}{x}\le 1$

$$-x^4\le x^4.\cos\frac{1}{x}\le x^4.$$

Taking limit through out the inequality, we have

$$0\le\underset{x\to 0}{Lt}\, x^4.\cos\frac{1}{x}\le 0$$

$$\underset{x\to 0}{Lt}\, x^4\cos\frac{1}{x}=0.$$

5 Evaluate $\underset{x\to 5}{Lt}\ 2x^2-3x+4$

$$\underset{x\to 5}{Lt}\ 2x^2-3x+4=50-15+4=39$$

6. Find $\underset{x\to 1}{Lt} f(x)\ \ if\ \ f(x)=\left|\begin{array}{ll} x+1, & if\ \ x\ne 1\\ 5, & if\ \ x=1\end{array}\right.$

Here $f(1)=5.$ But $\underset{x\to 1}{Lt} f(x)=\underset{x\to 1}{Lt} x+1=1+1=2$

7. Evaluate $\lim\limits_{x\to 0}|x|$.

Recall that $|x|=\begin{cases}x & if\ x\ge 0\\ -x & if\ x<0\end{cases}$

Since $|x|=x$ for $x>0$, we have, $\lim\limits_{x\to 0^+}|x|=\lim\limits_{x\to 0^+}x=0$

For $x<0$ we have $|x|=-x$ and so $\lim\limits_{x\to 0^-}|x|=\lim\limits_{x\to 0^-}(-x)=0$

Therefore $\lim\limits_{x\to 0}|x|=0.$

8. Prove that $\lim\limits_{x\to 0}\frac{|x|}{x}$ does not exist.

$$\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=\lim_{x\to 0^+}1=1$$

$$\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1$$

Since the right and left-hand limits are different, it follows that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

9 Find $\underset{x\to 2}{Lt}\left[3x-|x-2|\right]$ if it exists.

We know that

$$|x-2|=\begin{vmatrix}x-2, & if \quad x-2\geq 0\\ -(x-2), & if \quad x-2<0\end{vmatrix}$$

$$=\begin{vmatrix}x-2, & if \quad x\geq 2\\ 2-x, & if \quad x<2\end{vmatrix}$$

$$\therefore \underset{x\to 2^-}{Lt}\left[3x-|x-2|\right]=\underset{x\to 2^-}{Lt}\left[3x-2+x\right]=6-2+2=6$$

$$\therefore \underset{x\to 2^+}{Lt}\left[3x-|x-2|\right]=\underset{x\to 2^+}{Lt}\left[3x-x+2\right]=6-2+2=6$$

Since left and right limits are equal, $\underset{x\to 2}{Lt}\left[3x-|x-2|\right]=6.$

10 Find $\underset{x\to 0.5^-}{Lt}\frac{2x-1}{\left|2x^3-x^2\right|}$ if it exists.

Consider

$$\left|2x^3-x^2\right|=\left|x^2\right|.\left|2x-1\right|=x^2\,|2x-1|$$

$$But\ |2x-1|=\begin{vmatrix}2x-1, & if \quad 2x-1>0\\ -(2x-1), & if \quad 2x-1<0\end{vmatrix}$$

$$=\begin{vmatrix}2x-1, & if \quad x>0.5\\ -(2x-1), & if \quad x<0.5\end{vmatrix}$$

$$\therefore \left|2x^3-x^2\right|=\left|x^2\right|.\left|2x-1\right|=-x^2(2x-1)\ if\ x<0.5$$

$$\therefore \underset{x\to 0.5^-}{Lt}\frac{2x-1}{\left|2x^3-x^2\right|}=\underset{x\to 0.5^-}{Lt}\frac{2x-1}{-x^2(2x-1)}=-\frac{1}{(0.5)^2}=-4$$

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11 Show that $\underset{x\to 3}{Lt}\lceil x\rceil$ **does not exist where** $\lceil x\rceil$ **is the greatest integer function**

(i.e. largest integer less than or equal to x)

Here $\underset{x\to 3^{+}}{Lt}\lceil x\rceil=3 \quad and \quad \underset{x\to 3^{-}}{Lt}\lceil x\rceil=2$. Since left and right limits are not equal, limit does not exist.

Substitution theorem for limits

If $\underset{x\to a}{Lt} f(x)=c,$ then $\underset{x\to a}{Lt} g\left[f(x)\right]=\underset{y\to c}{Lt} g\left[y\right]$

To find the limit $\underset{x\to a}{Lt} g\left[f(x)\right]$, substitute y for $f(x)$ in $g\left[f(x)\right]$ and then find $\underset{y\to c}{Lt} g\left[y\right]$

Solved problems by applying substitution theorem

1. Find $\underset{x\to 0}{Lt}\sqrt{1-x^2}$

Put $y=1-x^2$

When $x\to 0,\ 1-x^2\to 1;\ \ i.e.\ \ y\to 1$

$$\therefore \underset{x\to 0}{Lt}\sqrt{1-x^2}=\underset{y\to 1}{Lt}\sqrt{y}=1$$

2. Check whether $\underset{x\to -3}{Lt}\dfrac{3x+9}{|x+3|}$ **exists.**

Let $x+3=y$ and hence $x\to -3\ means\ y\to 0.$

$$\underset{x\to -3}{Lt}\frac{3x+9}{|x+3|}=\underset{y\to 0}{Lt}\frac{3y}{|y|}$$

Consider $3\underset{y\to 0+}{Lt}\dfrac{y}{y}=3.1=3$

Also $3\underset{y\to 0-}{Lt}\dfrac{y}{-y}=3.(-1)=-3$

Since the right and left-hand limits are different, it follows that given limit does not exist.

3. Find $\underset{x\to \frac{\pi}{3}}{Lt}\cos\left(x+\dfrac{\pi}{6}\right)$

Put $y=x+\dfrac{\pi}{6}$

When $x\to\dfrac{\pi}{3},\ x+\dfrac{\pi}{6}=\dfrac{\pi}{3}+\dfrac{\pi}{6}\to\dfrac{\pi}{2};\ \ i.e.\ \ y\to\dfrac{\pi}{2}$

$$\therefore \underset{x\to \frac{\pi}{3}}{Lt}\cos\left(x+\frac{\pi}{6}\right)=\underset{y\to \frac{\pi}{2}}{Lt}\cos y=\cos\frac{\pi}{2}=0$$

.

4. Find $\mathop{Lt}\limits_{x \to \frac{\pi}{12}} \sqrt{\sin 2x}$

Put $y = 2x$

When

$x \to \frac{\pi}{12},\ 2x = 2\frac{\pi}{12} \to \frac{\pi}{6};\ \ i.e.\ \ y \to \frac{\pi}{6}$

$$\therefore \mathop{Lt}\limits_{x \to \frac{\pi}{12}} \sqrt{\sin 2x} = \mathop{Lt}\limits_{y \to \frac{\pi}{6}} \sqrt{\sin y}$$

Put $z = \sin y$

When

$y \to \frac{\pi}{6},\ \sin y = \sin\frac{\pi}{6} \to \frac{1}{2};\ \ i.e.\ \ z \to \frac{1}{2}$

$$\mathop{Lt}\limits_{x \to \frac{\pi}{12}} \sqrt{\sin 2x} = \mathop{Lt}\limits_{z \to \frac{1}{2}} \sqrt{z} = \sqrt{\frac{1}{2}}$$

5. Find $\mathop{Lt}\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

Put $y = \frac{\pi}{2} - x$

When $x \to \frac{\pi}{2},\ \frac{\pi}{2} - x \to 0;\ \ i.e.\ \ y \to 0$

$$\mathop{Lt}\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \mathop{Lt}\limits_{y \to 0} \left(\sec\left(\frac{\pi}{2} - y\right) - \tan\left(\frac{\pi}{2} - y\right)\right)$$

$$= \mathop{Lt}\limits_{y \to 0} \cos ecy - \cot y$$

$$= \mathop{Lt}\limits_{y \to 0} \frac{1}{\sin y} - \frac{\cos y}{\sin y}$$

$$= \mathop{Lt}\limits_{y \to 0} \frac{1 - \cos y}{\sin y}$$

$$= \mathop{Lt}\limits_{y \to 0} \frac{2\sin^2 \frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}}$$

$$= \mathop{Lt}\limits_{y \to 0} \frac{\sin\frac{y}{2}}{\cos\frac{y}{2}} = \frac{0}{1} = 0$$

Infinite Limits

Here we will discuss the limits whose value is infinity or minus infinity.

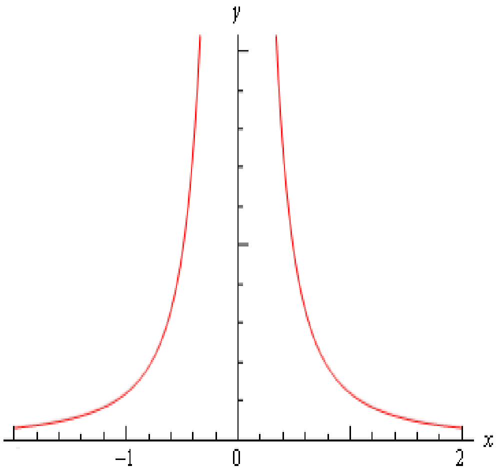
Consider the limit of the function $\frac{1}{x^2}$ *at* $x = 0$.

$$\underset{x\to 0^-}{Lt}\frac{1}{x^2}=\infty,\quad \text{since}$$

$$\frac{1}{(-0.000...1)^2}=\frac{1}{0.000000......1}=\infty$$

$$\underset{x\to 0}{Lt}\frac{1}{x^2}=\infty,\quad \text{since}\quad \frac{1}{(0.000...1)^2}=\frac{1}{0.000000......1}=\infty$$

Hence $\underset{x\to 0}{Lt}\frac{1}{x^2}=\infty$



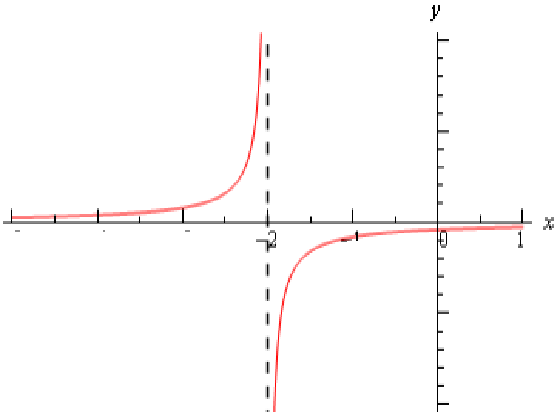
Solved problems on infinite limits

1. **Evaluate** (*i*) $\underset{x\to -2^+}{Lt}\frac{-1}{x+2}$ (*ii*) $\underset{x\to -2^-}{Lt}\frac{-1}{x+2}$ (*iii*) $\underset{x\to -2}{Lt}\frac{-1}{x+2}$

$$\underset{x\to -2^+}{Lt}\frac{-1}{x+2}=-\infty,\quad \text{since}\quad \frac{-1}{-1.9999999...+2}=\frac{-1}{0.000......1}=-\infty$$

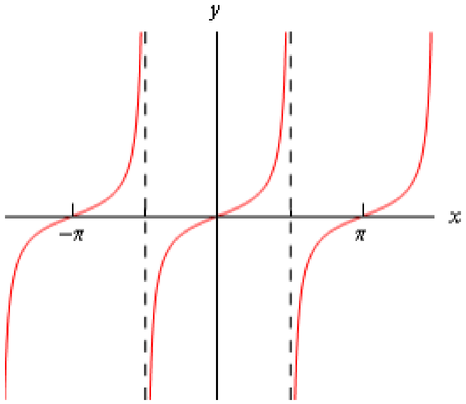
$$\underset{x\to -2^-}{Lt}\frac{-1}{x+2}=\infty,\quad \text{since}\quad \frac{-1}{-2.0000...1+2}=\frac{-1}{-0.000......1}=\infty$$

From the above $\underset{x\to -2}{Lt}\frac{-1}{x+2}$ does not exist.



2. **Evaluate** (*i*) $\underset{x\to \frac{\pi}{2}^+}{Lt}\tan x$ (*ii*) $\underset{x\to \frac{\pi}{2}^-}{Lt}\tan x$ (*iii*) $\underset{x\to \frac{\pi}{2}}{Lt}\tan x$

Let us draw the graph of $\tan x$



From the graph it is evident that $\underset{x \to \frac{\pi}{2}^{+}}{Lt} \tan x = -\infty$ and $\underset{x \to \frac{\pi}{2}^{-}}{Lt} \tan x = \infty$

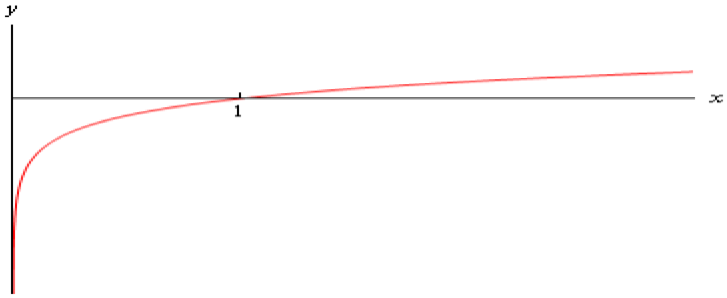
From the above $\underset{x \to \frac{\pi}{2}}{Lt} \tan x$ does not exist.

Vertical Asymptotes

1. If $\underset{x \to a^{+}}{Lt} f(x) = \infty$ *or* $\underset{x \to a^{+}}{Lt} f(x) = -\infty$, then the line $x = a$ is called vertical asymptote of the graph of f. (i.e. f has infinite right hand limit at a)

2. If $\underset{x \to a^{-}}{Lt} f(x) = \infty$ *or* $\underset{x \to a^{-}}{Lt} f(x) = -\infty$, then the line $x = a$ is called vertical asymptote of the graph of f. (i.e. f has infinite left hand limit at a)

Consider the graph of $\log_e x$



Here $\underset{x \to 0^{+}}{Lt} \log_e x = -\infty$. Hence $x = 0$ is the vertical asymptote of $\log_e x$.

1. Let $f(x) = \dfrac{x+2}{x^2-1}$. Find all the vertical asymptotes of the graph of f.

Here $\underset{x\to 1^{+}}{Lt}\ \frac{x+2}{x^2-1}=\underset{x\to 1^{+}}{Lt}\ \frac{x+2}{x+1}\frac{1}{x-1}=\infty$. Also $\underset{x\to -1^{-}}{Lt}\ \frac{x+2}{x^2-1}=\underset{x\to -1^{-}}{Lt}\ \frac{x+2}{x-1}\frac{1}{x+1}=\infty$

Therefore $x=1$ *and* $x=-1$ are the vertical asymptotes of the graph of f.

Limits at infinity

Here the limits of $f(x)$ when x approaches very large values in either positive or negative sense will be discussed.

$$\underset{x\to\infty}{Lt}\ f(x)\ \ or\ \ \underset{x\to-\infty}{Lt}\ f(x)$$

Results: 1. $\underset{x\to\infty}{Lt}\ \frac{1}{x^n}=0,$ where n is positive rational number.

2. $\underset{x\to-\infty}{Lt}\ \frac{1}{x^n}=0,$ where n is positive rational number and x^n is defined for $x<0$.

3. If $f(x)=a_nx^n+a_{n-1}x^{n-1}+....+a_0$ is a polynomial of degree n, then

$$\underset{x\to\infty}{Lt}\ f(x)=\underset{x\to\infty}{Lt}\ a_nx^n\ \ or\ \ \underset{x\to-\infty}{Lt}\ f(x)=\underset{x\to-\infty}{Lt}\ a_nx^n$$

Solved Problems

1. $\underset{x\to\infty}{Lt}\ 5x^3-2x^2-4x=\underset{x\to\infty}{Lt}\ x^3\left(5-\frac{2}{x}-\frac{4}{x^2}\right)$

$=5(\infty)$

$=\infty$

2. $\underset{x\to-\infty}{Lt}\ 5x^3-2x^2-4x=\underset{x\to-\infty}{Lt}\ x^3\left(5-\frac{2}{x}-\frac{4}{x^2}\right)$

$=5(-\infty)$

$=-\infty$

.

3. $\underset{x\to\infty}{Lt}\ \sin x$

We know that $\sin n\pi=0,\ \ \forall\ n$

Also $\sin(2n+1)\frac{\pi}{2}=\pm1,\ \ \forall\ n$

Therefore $\sin x$ never exceeds 1 even for large values of n.

$\therefore\ \underset{x\to\infty}{Lt}\ \sin x$ does not exist.

4. $\underset{x\to\infty}{Lt}\ \frac{x-1}{x+1}$

$$\underset{x\to\infty}{Lt}\ \frac{x-1}{x+1}=\underset{x\to\infty}{Lt}\ \frac{x\left(1-\frac{1}{x}\right)}{x\left(1+\frac{1}{x}\right)}$$

$$=\underset{y\to 0}{Lt}\ \frac{(1-y)}{(1+y)},\qquad y=\frac{1}{x}$$

$$=\frac{1-0}{1+0}=1$$

.

5 Evaluate $\underset{x\to\infty}{Lt}\ \dfrac{\sqrt{2x^2+1}}{2-3x}$

$$\underset{x\to\infty}{Lt}\ \frac{\sqrt{2x^2+1}}{2-3x}=\underset{x\to\infty}{Lt}\ \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(\frac{2}{x}-3\right)}$$

$$=\underset{x\to\infty}{Lt}\ \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(\frac{2}{x}-3\right)}$$

$$=\underset{x\to\infty}{Lt}\ \frac{x\sqrt{2+\frac{1}{x^2}}}{x\left(\frac{2}{x}-3\right)}$$

$$=\underset{x\to\infty}{Lt}\ \frac{\sqrt{2+0}}{(0-3)}$$

$$=\frac{\sqrt{2}}{-3}$$

6. Evaluate $\underset{x\to-\infty}{Lt}\ \dfrac{\sqrt{2x^2+1}}{2-3x}$

$$\underset{x\to-\infty}{Lt}\ \frac{\sqrt{2x^2+1}}{2-3x}=\underset{x\to-\infty}{Lt}\ \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(\frac{2}{x}-3\right)}$$

$$=\underset{x\to-\infty}{Lt}\ \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(\frac{2}{x}-3\right)}$$

$$=\underset{x\to-\infty}{Lt}\ \frac{-x\sqrt{2+\frac{1}{x^2}}}{x\left(\frac{2}{x}-3\right)}$$

$$=\underset{x\to-\infty}{Lt}\ \frac{-\sqrt{2+0}}{(0-3)}$$

$$=\frac{\sqrt{2}}{3}$$

.

7. Evaluate $\underset{x\to\infty}{Lt}\ \dfrac{3e^{4x}-2e^{-3x}}{5e^{4x}+4e^{2x}-2e^{-x}}$

Since limit is ∞, the largest exponent In the denominator should be taken out as common term

$$\underset{x\to\infty}{Lt}\ \frac{3e^{4x}-2e^{-3x}}{5e^{4x}+4e^{2x}-2e^{-x}}$$

$$=\underset{x\to\infty}{Lt}\ \frac{e^{4x}\left(3-2e^{-7x}\right)}{e^{4x}\left(5+4e^{-2x}-2e^{-5x}\right)}$$

$$=\underset{x\to\infty}{Lt}\ \frac{\left(3-2e^{-7x}\right)}{\left(5+4e^{-2x}-2e^{-5x}\right)}$$

$$=\frac{3-0}{5+0}$$

$$=\frac{3}{5}$$

8. Evaluate $\underset{x\to-\infty}{Lt}\ \dfrac{3e^{4x}-2e^{-3x}}{5e^{4x}+4e^{2x}-2e^{-x}}$

Since limit is $-\infty$, the lowest exponent In the denominator should be taken out as common term

$$\underset{x\to-\infty}{Lt}\ \frac{3e^{4x}-2e^{-3x}}{5e^{4x}+4e^{2x}-2e^{-x}}$$

$$=\underset{x\to-\infty}{Lt}\ \frac{e^{-x}\left(3e^{5x}-2e^{-2x}\right)}{e^{-x}\left(5e^{5x}+4e^{3x}-2\right)}$$

$$=\underset{x\to-\infty}{Lt}\ \frac{\left(3e^{5x}-2e^{-2x}\right)}{\left(5e^{5x}+4e^{3x}-2\right)}$$

$$=\frac{0-\infty}{0+0-2}$$

$$=\infty$$

9. Evaluate $\underset{x\to-\infty}{Lt}\ e^{x}$

Put $y=-x$

When $x\to-\infty,\ then\ y\to\infty$

$$\therefore \underset{x\to-\infty}{Lt}\ e^{x}=\underset{y\to\infty}{Lt}\ e^{-y}$$

But $e^{y}\geq 1+y$ for all positive y.

Then $e^{-y}=\frac{1}{y}\leq\frac{1}{1+y}$

But $\underset{y\to\infty}{Lt}\ \frac{1}{1+y}=0$

Therefore $\underset{y\to\infty}{Lt}\ e^{-y}=0$

$$\therefore \underset{x\to-\infty}{Lt}\ e^{x}=0$$

10. Evaluate $\underset{x\to\infty}{Lt}\ e^{\left(2-3x-4x^{2}\right)}$

Consider

$$\underset{x\to\infty}{Lt}\ \left(2-3x-4x^{2}\right)=\underset{x\to\infty}{Lt}\ \left(-4x^{2}\right)=-4(\infty)=-\infty$$

Therefore

$$\underset{x\to\infty}{Lt}\ e^{\left(2-3x-4x^{2}\right)}=e^{-\infty}=0$$

9. Evaluate $\underset{x\to-\infty}{Lt}\ e^{\left(2-3x+4x^{2}\right)}$

Consider

$$\underset{x\to-\infty}{Lt}\ \left(2-3x+4x^{2}\right)=\underset{x\to-\infty}{Lt}\ \left(4x^{2}\right)=4(\infty)=\infty$$

Therefore

$$\underset{x\to-\infty}{Lt}\ e^{\left(2-3x+4x^{2}\right)}=e^{\infty}=\infty$$

.

Horizontal Asymptote

The function $f(x)$ will have a horizontal asymptote at $y=l$ if

$$\underset{x\to\infty}{Lt}\ f(x)=l\ \ or\ \ \underset{x\to-\infty}{Lt}\ f(x)=l$$

Consider $\underset{x\to\infty}{Lt}\ \frac{x^{2}-2x+3}{x^{3}+2x^{2}-x}=\underset{x\to\infty}{Lt}\ \frac{x^{3}\left(\frac{1}{x}-\frac{2}{x^{2}}+\frac{3}{x^{3}}\right)}{x^{3}\left(1+\frac{1}{x}-\frac{1}{x^{2}}\right)}=\underset{x\to\infty}{Lt}\ \frac{\left(\frac{1}{x}-\frac{2}{x^{2}}+\frac{3}{x^{3}}\right)}{\left(1+\frac{1}{x}-\frac{1}{x^{2}}\right)}=\frac{0}{1}=0$

Therefore $y=0$ is the horizontal asymptote to the given function.

Note: If $f(x)\leq g(x)$, then $\underset{x\to a}{Lt} f(x)\leq\underset{x\to a}{Lt} g(x)$

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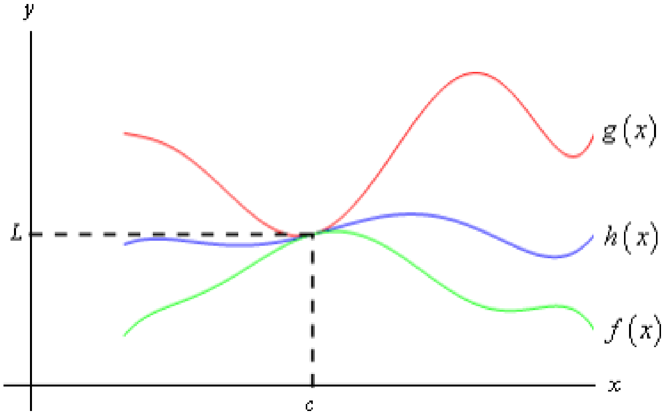
Squeeze Theorem: If $f(x) \le g(x) \le h(x)$ and $\underset{x \to a}{Lt} f(x) = l$, $\underset{x \to a}{Lt} h(x) = l$, then $\underset{x \to a}{Lt} g(x) = l$.

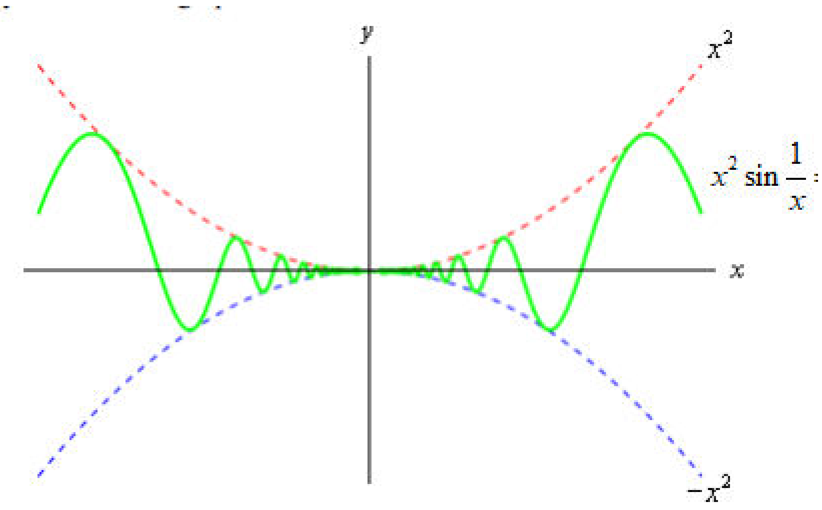


Use squeeze theorem, to show that $\underset{x \to 0}{Lt} x^2 \sin\frac{1}{x} = 0$.

We know that $-1 \le \sin\frac{1}{x} \le 1$. Multiply throughout by x^2.

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

Here $\underset{x \to 0}{Lt} -x^2 = 0$ *and* $\underset{x \to 0}{Lt} x^2 = 0$. Hence by squeeze theorem $\underset{x \to 0}{Lt} x^2 \sin\frac{1}{x} = 0$.



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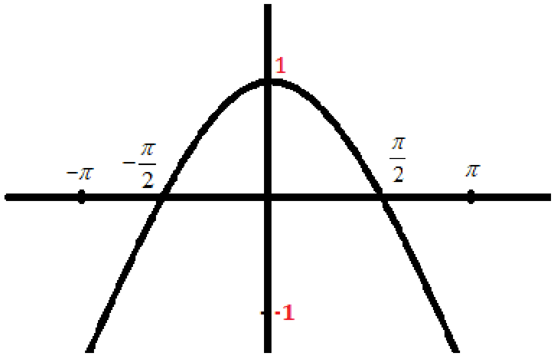
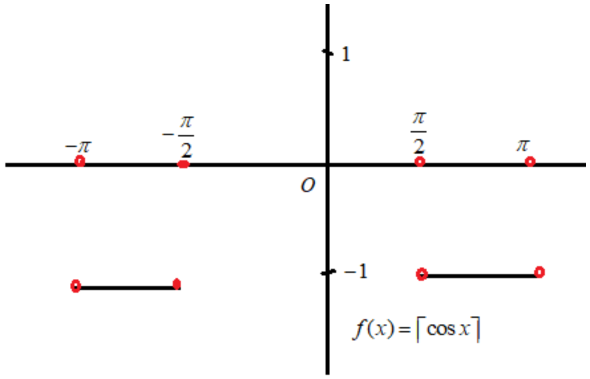
1. **Let** $f(x)=\lceil \cos x \rceil,\ -\pi \le x \le \pi$

(a) Sketch the graph of $f(x)$**.** **(b) Find** $\underset{x\to 0}{Lt} f(x)$

(c) Find $\underset{x\to \pi/2^{-}}{Lt} f(x)$ **(d) For what value of** a **does** $\underset{x\to a}{Lt} f(x)$ **exist?**



$f(x)=\cos x$



Since $-1 \le x < 0$ *in* $\left[-\pi,\frac{\pi}{2}\right)$, we have $f(x)=\lceil \cos x \rceil=-1$ in the interval $\left[-\pi,\frac{\pi}{2}\right)$

Since $0 \le x < 1$ *in* $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$, we have $f(x)=\lceil \cos x \rceil=0$ in $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$

Since $-1 \le x < 0$ *in* $\left(\frac{\pi}{2},\pi\right]$, we have $f(x)=\lceil \cos x \rceil=-1$ in the interval $\left(\frac{\pi}{2},\pi\right]$

Since $\underset{x\to 0^{-}}{Lt} f(x)=0=\underset{x\to 0^{+}}{Lt} f(x)$, we have $\underset{x\to 0}{Lt} f(x)=0.$

Also $\underset{x\to \pi/2^{-}}{Lt} f(x)=0$ *and* $\underset{x\to \pi/2^{+}}{Lt} f(x)=-1$

From the diagram, it is observed that $\underset{x\to a}{Lt} f(x)$ exists for all a in $(-\pi,\pi)$ except at $x=\pm\frac{\pi}{2}$.

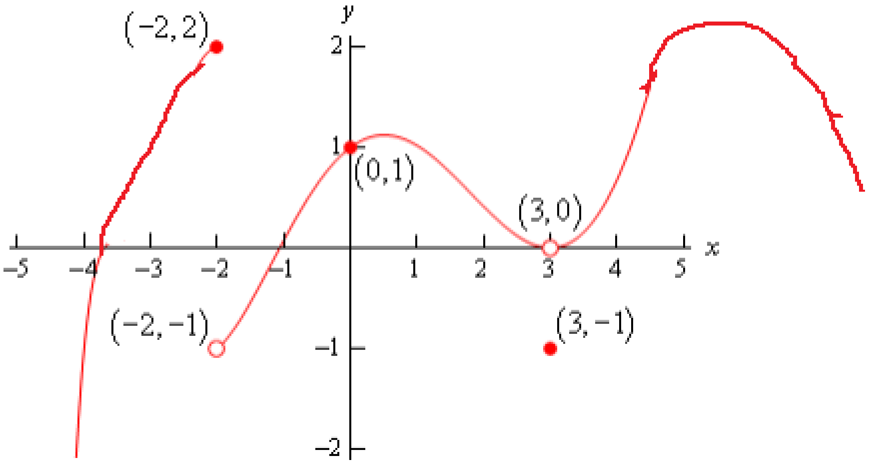
Continuity

Definition: A function $f(x)$ is said to be continuous at a point $'a'$ in its domain if $\underset{x \to a}{Lt} f(x) = f(a)$.

A function $f(x)$ is discontinuous at a point $'a'$ in its domain if it is not continuous at a.

Result: A function $f(x)$ is continuous at a point $'a'$ then $\underset{x \to a^+}{Lt} f(x) = \underset{x \to a^-}{Lt} f(x) = \underset{x \to a}{Lt} f(x) = f(a)$.

Example: From the given graph, determine whether $f(x)$ is continuous at $x = -2,\ 0,\ 3$.



Let $x = -2$.

Here $f(-2) = -2$.

$\underset{x \to -2^-}{Lt} f(x) = 2$

$\underset{x \to -2^+}{Lt} f(x) = -1$

$\therefore \underset{x \to -2}{Lt} f(x)$ does not exist

So $f(x)$ is not continuous and it is called jump discontinuity.

Let $x = 0$.

Here $f(0) = 1$.

$\underset{x \to 0^+}{Lt} f(x) = 1$

$\underset{x \to 0^-}{Lt} f(x) = 1$

$\therefore \underset{x \to 0}{Lt} f(x) = 1$

So $f(x)$ is continuous.

Let $x = 3$.

Here $f(3) = -1$.

$\underset{x \to 3^+}{Lt} f(x) = 0$

$\underset{x \to 3^-}{Lt} f(x) = 0$

So $f(x)$ is not continuous and it is called removable discontinuity.

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Where are the following functions discontinuous?

$$(i)\ f(x)=\frac{x^2+x-1}{x-1} \qquad (ii)\ f(x)=\begin{vmatrix}\frac{1}{x^2},\ at\ x\neq 0\\ 1,\ at\ x=0\end{vmatrix} \qquad (iii)\ f(x)=\begin{vmatrix}\frac{x^2-3x+2}{x-2}, & at\ x\neq 2\\ 2\qquad , & at\ x=2\end{vmatrix}$$

(i) Here $f(1)$ is not defined. Hence $f(x)$ is discontinuous at $x=1$.

(ii) Here $f(0)$ is defined. But $\underset{x\to 0}{Lt}\frac{1}{x^2}$ does not exist. $\therefore$ $f(x)$ is discontinuous at $x=0$.

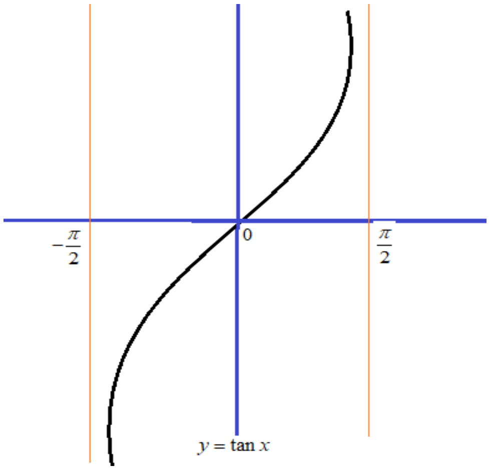
(iii) Here $f(2)=2$ is defined. But $\underset{x\to 2}{Lt}\frac{x^2-3x+2}{x-2}=\underset{x\to 2}{Lt}\left(\frac{(x-1)(x-2)}{(x-2)}\right)=\underset{x\to 2}{Lt}(x-1)=1$

Here $\underset{x\to 2}{Lt}f(x)\neq f(2)$. $\therefore$ $f(x)$ is discontinuous at $x=2$.

Discuss the continuity of $f(x)=\tan x$

We know that $f(x)=\frac{\sin x}{\cos x}$ is continuous at all points except at $\cos x=0$. This happens when $x=\pm(2n+1)\frac{\pi}{2}$, n is integer.

Therefore $f(x)=\tan x$ is discontinuous at $x=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\ldots$



Discuss the continuity of $f(x)=\frac{\log x+\tan^{-1}x}{x^2-1}$

We know that $\log x$ is continuous for $x>0$ and $\tan^{-1}x$ is continuous in $(-\infty,\infty)$ and hence Nr is continuous in $(0,\infty)$. Also x^2-1 is continuous everywhere. Therefore $f(x)$ is discontinuous at $x^2-1=0$. *i.e.* $x=\pm 1$. i.e. $f(x)$ continuous at $(0,1)$ *and* $(1,\infty)$.

To verify whether a function is continuous at a point, the following theorems may be useful.

Theorem: If f and g are continuous at a point a and c is a constant, then the following are also continuous at the point a: $f+g,\ f-g,\ fg,\ cf,\ \frac{f}{g}\ if\ g(a)\neq 0$

Theorem: Any polynomial is continuous in everywhere in R.

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Theorem: Any rational function is continuous whenever it is defined.

Results: All trigonometric, exponential and logarithmic functions are continuous in its domain.

Theorem: If $f(x)$ **is continuous at** $'b'$ **and** $\underset{x \to a}{Lt}\, g(x) = b$ **then** $\underset{x \to a}{Lt}\, f[g(x)] = f(b)$ **.**

OR $\underset{x \to a}{Lt}\, f[g(x)] = f\left[\underset{x \to a}{Lt}\, g(x)\right] = f\left[g(a)\right]$

i.e. $f \circ g$ is continuous at a .(A continuous function of a continuous function is continuous)

Where is the function $h(x) = \sin\left(x^2\right)$ **continuous?**

Let $g(x) = x^2 \; and \; f(x) = \sin x, \; hence \; h(x) = \left(f \circ g\right)(x) = \sin\left(x^2\right)$

Here $g(x) \, and \; f(x)$ is continuous everywhere and hence $\left(f \circ g\right)(x)$ is continuous.

Definition: A function $f(x)$ **is said to be continuous from the right at a point** $'a'$ **if** $\underset{x \to a^+}{Lt}\, f(x) = f(a).$

A function $f(x)$ **is said to be continuous from the left at a point** $'a'$ **if** $\underset{x \to a^-}{Lt}\, f(x) = f(a).$

Solved Problems

1 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ **is continuous in** $[-1,1]$**.**

Suppose $-1 < a < 1$ then, $\underset{x \to a}{Lt}\, f(x) = \underset{x \to a}{Lt}\, 1 - \sqrt{1 - x^2} = 1 - \sqrt{1 - a^2} = f(a).$

Hence f is continuous in $-1 < a < 1$.

Also $\underset{x \to -1^+}{Lt}\, f(x) = \underset{x \to -1^+}{Lt}\, 1 - \sqrt{1 - x^2} = 1 = f(-1). \;\; \therefore \; f$ is continuous from the right at -1

And $\underset{x \to 1^-}{Lt}\, f(x) = \underset{x \to 1^-}{Lt}\, 1 - \sqrt{1 - x^2} = 1 = f(1). \;\; \therefore \; f$ is continuous from the left at 1.

Hence f is continuous in $[-1,1]$.

2 $f(x) = \; x$

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The greatest integer function $f(x) = \ x\ $ is discontinuous at all integers. For example $\underset{x\to 2^-}{Lt}\ x\ = 1$ *and* $\underset{x\to 2^+}{Lt}\ x\ = 2$ *and* $f(2) = \ 2\ = 2$.

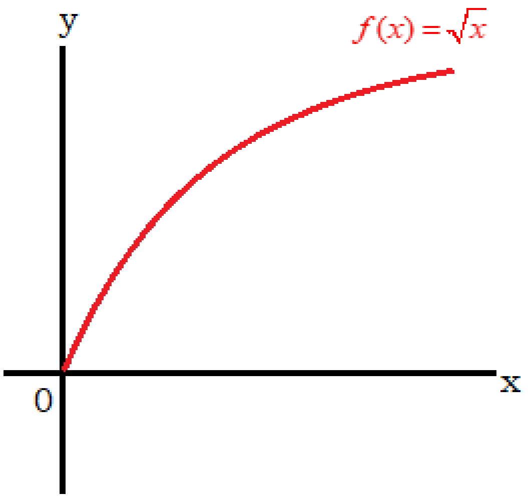
3 Show that $f(x)=\sqrt{x}$ is continuous from right at 0.

Here $f(x)=\sqrt{x}$ is not defined for $x<0$. Therefore it is not continuous at 0.

Here $f(0)=\sqrt{0}=0$

and $\underset{x\to 0^+}{Lt} f(x)=\sqrt{0}=0$.

Hence it is continuous from right at 0.



.

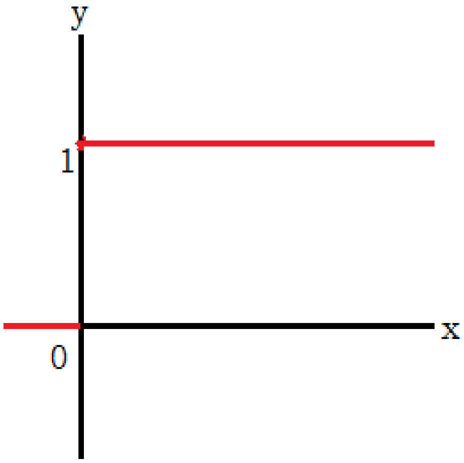
4 Show that $f(x)=\begin{cases}0, & for\ \ x<0\\ 1, & for\ \ x\ge 0\end{cases}$ is continuous from the right at 0 but not continuous from the left at 0.

Here $f(0)=1$

and $\underset{x\to 0^+}{Lt} f(x)=1$.

Hence it is continuous from right at 0.

But $\underset{x\to 0^-}{Lt} f(x)=0$. Hence it is not continuous from left at 0.



5 Show that $f(x)=\sqrt{1-x^2}$ is continuous in $[-1,1]$.

Since $1-x^2\ge 0$ if and only if $-1\le x\le 1$. Hence the domain of $f(x)$ is $[-1,1]$.

Since $\underset{x\to a}{Lt} f(x)=\underset{x\to a}{Lt}\sqrt{1-x^2}=\sqrt{1-a^2}=f(a)$ *for* $-1<a<1$, $f(x)$ is continuous in $-1<a<1$

Also $\underset{x\to 1^-}{Lt} f(x)=\underset{x\to 1^-}{Lt}\sqrt{1-x^2}=\sqrt{1-1^2}=0=f(1)$. Hence $f(x)$ is continuous from left at 1.

and $\underset{x \to -1^+}{Lt} f(x) = \underset{x \to -1^+}{Lt} \sqrt{1-x^2} = \sqrt{1-(-1)^2} = 0 = f(-1)$. Hence $f(x)$ is continuous from right at -1.

Hence $f(x)$ is continuous in $[-1,1]$.

6 For what value of c is the function $f(x)$ continuous in $(-\infty,\infty)$ where $f(x) = \begin{vmatrix} cx^2+2x, & x<2 \\ x^3-cx, & x \geq 2 \end{vmatrix}$

$$\underset{x \to 2^-}{Lt} f(x) = \underset{x \to 2^-}{Lt} cx^2 + 2x = 4c+4 \text{ and } \underset{x \to 2^+}{Lt} f(x) = \underset{x \to 2^+}{Lt} x^3 - cx = 8-2c$$

Given that $f(x)$ is continuous at $x = 2$. Hence

$$\underset{x \to 2^-}{Lt} f(x) = \underset{x \to 2^+}{Lt} f(x)$$

$$4c+4 = 8-2c$$

$$6C = 4$$

$$C = \frac{2}{3}$$

Therefore $f(x)$ continuous in $(-\infty,\infty)$ when $C = \frac{2}{3}$

7 If $f(x) = \begin{vmatrix} 1, & if & x \leq 3 \\ ax+b, & if & 3<x<5 \\ 7, & if & 5 \geq x \end{vmatrix}$**, determine the values of a and b, so that it is continuous.**

Given that $f(x)$ is continuous in the domain. We discuss the continuity at $x = 3$ & $x = 5$.

$$\underset{x \to 3^-}{Lt} f(x) = \underset{x \to 3^-}{Lt} 1 = 1 \text{ and } \underset{x \to 3^+}{Lt} f(x) = \underset{x \to 3^+}{Lt} ax+b = 3a+b$$

Since $\underset{x \to 3^-}{Lt} f(x) = \underset{x \to 3^+}{Lt} f(x)$, we have $3a+b = 1$.....(1)

$$\underset{x \to 5^+}{Lt} f(x) = \underset{x \to 5^+}{Lt} 7 = 7 \text{ and } \underset{x \to 5^-}{Lt} f(x) = \underset{x \to 5^-}{Lt} ax+b = 5a+b$$

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Since $\underset{x\to 5^-}{Lt} f(x) = \underset{x\to 5^+}{Lt} f(x)$, we have $5a+b=7.....(2)$

Solving (1) & (2), we have $a=3, \quad b=-8.$

8 Find the values of a and b that make f continuous everywhere on the real line.

$$f(x)=\left|\begin{array}{ll}\dfrac{x^2-4}{x-2}, & if \quad x<2 \\ ax^2-bx+3, & if \quad 2\le x<3 \\ 2x-a+b, & if \quad x\ge 3\end{array}\right.$$

Given that $f(x)$ is continuous in the domain. We discuss the continuity at $x=2$ & $x=3$.

$$\underset{x\to 2^-}{Lt} f(x) = \underset{x\to 2^-}{Lt} \frac{x^2-4}{x-2} = \underset{x\to 2^-}{Lt} \frac{(x-2)(x+2)}{(x-2)} = 2+2=4$$

$$\underset{x\to 2^+}{Lt} f(x) = \underset{x\to 2^+}{Lt} ax^2-bx+3 = 4a-2b+3$$

Since left and right limit at $x=2$ are equal, we have $4a-2b+3=4.........(1)$

$$\underset{x\to 3^-}{Lt} f(x) = \underset{x\to 3^-}{Lt} ax^2-bx+3 = 9a-3b+3$$

$$\underset{x\to 3^+}{Lt} f(x) = \underset{x\to 3^+}{Lt} 2x-a+b = 6-a+b$$

Since left and right limit at $x=3$ are equal, we have $9a-3b+3=6-a+b.........(2)$

From (1) and (2), we have

$$\begin{aligned} 4a-2b&=1.........(3) \\ 10a-4b&=3.........(4) \\ -2\times(3) \Rightarrow \quad -8a+4b&=-2........(5) \end{aligned}$$

Adding (4) and (5), $2a=1 \quad i.e. \ a=\dfrac{1}{2}$

From (3), $4.\dfrac{1}{2}-2b=1 \quad i.e. \quad 2b=2-1 \quad i.e. \ b=\dfrac{1}{2}$

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9 **For what values of a and b, is** $f(x) = \left| \begin{matrix} -2, & x \le -1 \\ ax-b, & -1 < x < 1 \\ 3, & x \ge 1 \end{matrix} \right.$ **continuous at every x?**

Given that $f(x)$ is continuous in the domain. We discuss the continuity at $x=-1$ & $x=1$.

$$\underset{x \to -1^-}{Lt} f(x) = \underset{x \to -1^-}{Lt} 2 = 2$$

$$\underset{x \to -1^+}{Lt} f(x) = \underset{x \to -1^+}{Lt} ax-b = -a-b$$

Since left and right limit at $x=-1$ are equal, we have $-a-b = 2$.........(1)

$$\underset{x \to 1^-}{Lt} f(x) = \underset{x \to 1^-}{Lt} ax-b = a-b$$

$$\underset{x \to 1^+}{Lt} f(x) = \underset{x \to 1^+}{Lt} 3 = 3$$

Since left and right limit at $x=1$ are equal, we have $a-b=3$.........(2)

Adding (1) and (2), $-2b = 5$ $i.e.$ $b = -\frac{5}{2}$

From (2), $a = 3+b = 3-\frac{5}{2} = \frac{1}{2}$

10 **Show that the function** $f(x) = \left| \begin{matrix} x^4 \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{matrix} \right.$ **is continuous in** $(-\infty,\infty)$**.**

Here $f(x)$ is the product of polynomial and composite of trigonometric and rational function, it is continuous in $(-\infty,0) \cup (0,\infty)$.

We know that $-1 \le \sin\frac{1}{x} \le 1$ and hence $-x^4 \le \sin\frac{1}{x} \le x^4$

Also $\underset{x \to 0}{Lt}\left(-x^4\right) = 0$ and $\underset{x \to 0}{Lt}\left(x^4\right) = 0$.

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Therefore by Squeeze theorem, $\underset{x\to 0}{Lt} x^4 \sin\frac{1}{x} = 0 = f(0)$

Therefore $f(x)$ is continuous at 0 and hence $f(x)$ is continuous at $(-\infty,\infty)$.

11. Check whether $f(x)=\begin{cases} \frac{x^2+4x-12}{x^2-2x}, & x\neq 2 \\ 6, & x=2 \end{cases}$ **is continuous or not at** $x=2$.

Given that $f(2)=6$. Now let us find the left and right limit at $x=2$.

$$\underset{x\to 2^-}{Lt} f(x) = \underset{x\to 2^-}{Lt} \frac{x^2+4x-12}{x^2-2x} = \underset{x\to 2^-}{Lt} \frac{(x-2)(x+6)}{x(x-2)} = \underset{x\to 2^-}{Lt} \frac{(x+6)}{x} = \frac{8}{2} = 4$$

$$\underset{x\to 2^+}{Lt} f(x) = \underset{x\to 2^+}{Lt} \frac{x^2+4x-12}{x^2-2x} = \underset{x\to 2^+}{Lt} \frac{(x-2)(x+6)}{x(x-2)} = \underset{x\to 2^+}{Lt} \frac{(x+6)}{x} = \frac{8}{2} = 4$$

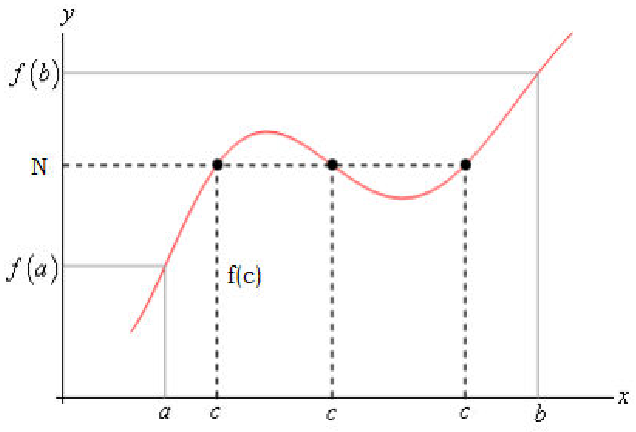
Here $\underset{x\to 2^-}{Lt} f(x) = \underset{x\to 2^+}{Lt} f(x) \neq f(2)$.

$\therefore$ $f(x)$ is not continuous at $x=2$.

The Intermediate Value Theorem

Suppose $f(x)$ is continuous in the closed interval $[a,b]$ and let N be any number between $f(a)$ *and* $f(b)$, where $f(a)\neq f(b)$. Then there exists a number c in (a,b) such that $f(c)=N$.

(i.e. a continuous function takes on all values between $f(a)$ *and* $f(b)$). It is used for locating the roots of an equation.



Show that there is a root of the equation $x^3+x^2+x-2=0$ **lies between** -1 *and* 1.

Let $f(x)=x^3+x^2+x-2$, which is continuous. Assume that $a=-1,\ b=1$.

Here $f(-1)=-1+1-1-2=-3$ *and* $f(2)=8+4+2-2=12$.

Let $N=0$, lies between – 3 and 12.

Therefore by intermediate value theorem there exists a number c such that $f(c)=0$ and c is the root of the equation.

Show that there is a root of the equation $4x^3-6x^2+3x-2=0$ **lies between** 1 *and* 2.

Let $f(x)=4x^3-6x^2+3x-2$, which is continuous. Assume that $a=1, b=2$.

Here $f(1)=4-6+3-2=-1$ *and* $f(2)=32-24+6-2=12$.

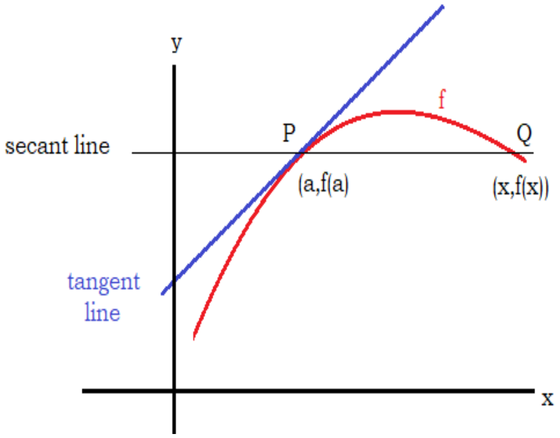
Let $N=0$, lies between – 1 and 12.

Therefore by intermediate value theorem there exists a number c such that $f(c)=0$ and c is the root of the equation.

Tangent Lines

Let $y=f(x)$ be a curve and $P[a,f(a)]$ & $Q[x,f(x)]$ be any two points on it. Then the slope of the secant line PQ is $\frac{f(x)-f(a)}{x-a}$.

Let Q approach P along the curve by letting x approach a. Then we get a tangent at P with slope m.



Therefore the tangent to the curve $y=f(x)$ at $P[a,f(a)]$ is a line through P with slope

$$m=\underset{x\to a}{Lt}\ \frac{f(x)-f(a)}{x-a}.$$

Therefore the equation tangent of f at $P[a,f(a)]$ is $y-f(a)=m(x-a)$

Let $h=x-a$ *and* $x=a+h$ then $m=\underset{h\to 0}{Lt}\ \frac{f(a+h)-f(a)}{h}$.

Solved Problems

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1 Show that there is a tangent to $f(x)=x^3-1$ at $(1,0)$ and find its equation.

We know that $m=\underset{x\to 1}{Lt}\dfrac{f(x)-f(1)}{x-1}$

$$=\underset{x\to 1}{Lt}\frac{\left(x^3-1\right)-(1-1)}{x-1}$$

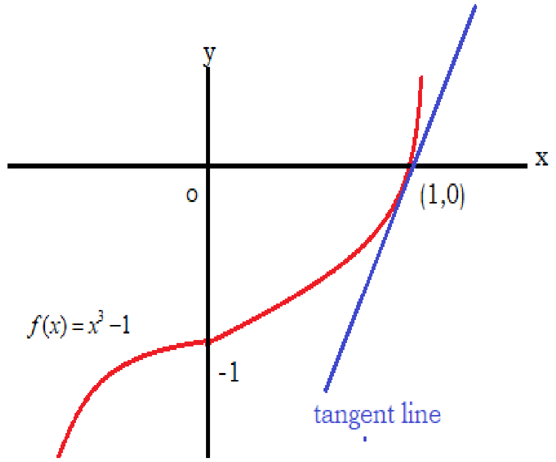
$$=\underset{x\to 1}{Lt}\frac{(x-1)\left(x^2+x+1\right)}{x-1}$$

$$=1+1+1=3$$

Therefore the equation tangent of f at $P[1,0]$ is

$y-0=3(x-1)$

$y=3x-3$



.

2 Find the equation of the tangent line to the parabola $y=x^2$ at $(1,1)$.

Slope of the tangent is

$$m=\underset{x\to a}{Lt}\frac{f(x)-f(a)}{x-a}$$

$$=\underset{x\to 1}{Lt}\frac{f(x)-f(1)}{x-1}=\underset{x\to 1}{Lt}\frac{x^2-1}{x-1}$$

$$=\underset{x\to 1}{Lt}\frac{(x-1)(x+1)}{x-1}=1+1=2$$

The equation of tangent at $\left(x_1,y_1\right)$ is

$y-y_1=m\left(x-x_1\right)$

Therefore the equation of tangent at $(1,1)$ is

$y-1=2\left(x-1\right)$

$i.e.\ \ y=2x-1$

.

3 Find the equation tangent to the parabola $f(x)=x^2-5x+5$ at a point $(3,-1)$.

Slope of the tangent is

$$m=\underset{h\to 0}{Lt}\frac{f(a+h)-f(a)}{h}\quad where\ \ f(x)=x^2-5x+5$$

$$=\underset{h\to 0}{Lt}\frac{(a+h)^2-5(a+h)+5-(a^2-5a+5)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{a^2+h^2+2ah-5a-5h+5-a^2+5a-5}{h}$$

$$=\underset{h\to 0}{Lt}\frac{h^2+2ah-5h}{h}=\underset{h\to 0}{Lt}\frac{h(h+2a-5)}{h}=2a-5$$

Here $a=3$ and hence $m=2a-5=6-5=1$

Therefore the equation of tangent at $(3,-1)$ is

$y+1=1\left(x-3\right)\quad i.e.\quad y=x-2$

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4 Find the equation of the tangent line to the hyperbola $y = \frac{3}{x}$ at $(3,1)$.

Slope of the tangent is

$$m = \underset{h \to 0}{Lt} \frac{f(a+h)-f(a)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{f(3+h)-f(3)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\frac{3}{3+h}-1}{h}$$

$$= \underset{h \to 0}{Lt} \frac{3-3-h}{(3+h)h}$$

$$= \underset{h \to 0}{Lt} \frac{-1}{(3+h)} \quad = -\frac{1}{3}$$

The equation of tangent at $\left(x_1, y_1\right)$ is

$$y - y_1 = m\left(x - x_1\right)$$

Therefore the equation of tangent at $(3,1)$ is

$$y - 1 = -\frac{1}{3}\left(x-3\right)$$

$$3y - 3 = -x + 3 \quad i.e. \quad 3y = -x + 6$$

.

Note: If f is continuous at a point a and $m = \underset{x \to a}{Lt} \frac{f(x)-f(a)}{x-a} = \pm\infty$, then the vertical line $x = a$ is the tangent.

Note: If f is continuous at a point a and $m = \underset{x \to a}{Lt} \frac{f(x)-f(a)}{x-a}$ does not exist, then f has no tangent at $x = a$.

Find the equation tangent to $f(x) = x^{\frac{1}{3}}$ at $(0,0)$.

Obviously f is continuous at a point 0 and

$$m = \underset{x \to 0}{Lt} \frac{f(x)-f(0)}{x-0}$$

$$= \underset{x \to 0}{Lt} \frac{x^{\frac{1}{3}}-0}{x-0} = \underset{x \to 0}{Lt} x^{-\frac{2}{3}} = \underset{x \to 0}{Lt} \frac{1}{x^{\frac{2}{3}}} = \infty$$

$x = 0$ is the tangent line to the given f.

Show that there is no tangent to $f(x) = |x|$ at $(0,0)$.

The right hand limit at 0 is

$$\underset{x \to 0^+}{Lt} \frac{f(x)-f(0)}{x-0} = \underset{x \to 0^+}{Lt} \frac{|x|-0}{x-0} = \underset{x \to 0^+}{Lt} \frac{x-0}{x-0} = 1$$

The left hand limit at 0 is

$$\underset{x \to 0^-}{Lt} \frac{f(x)-f(0)}{x-0} = \underset{x \to 0^-}{Lt} \frac{|x|-0}{x-0} = \underset{x \to 0^+}{Lt} \frac{-x-0}{x-0} = -1$$

Hence $\underset{x \to 0}{Lt} \frac{f(x)-f(0)}{x-0}$ doen not exist and is not

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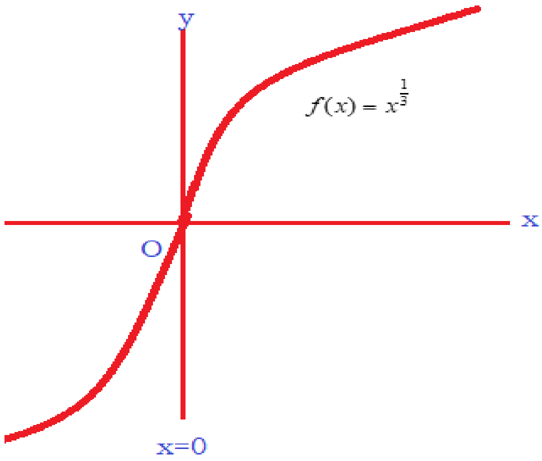
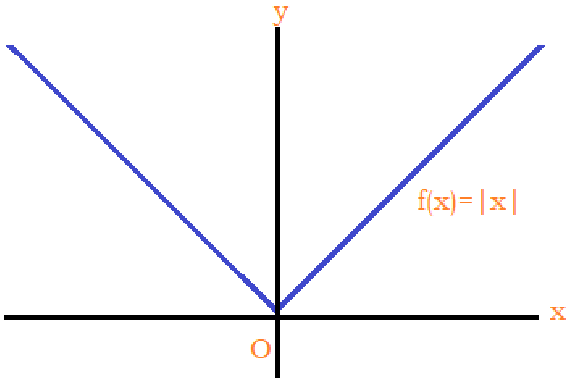
infinite. Therefore f has no tangent at (0,0).





The Derivative

Let a be a number in the domain of a function f. The derivative of a function f at a point a, denoted by $f'(a)$ is $f'(a) = \underset{x \to a}{Lt} \dfrac{f(x)-f(a)}{x-a}$.

Note:

1 The equivalent form of $f'(a)$ is $f'(a) = \underset{h \to 0}{Lt} \dfrac{f(a+h)-f(a)}{h}$.

2 The derivative as a function is given by $f'(x) = \underset{t \to x}{Lt} \dfrac{f(t)-f(x)}{t-x}$ or $f'(x) = \underset{h \to 0}{Lt} \dfrac{f(x+h)-f(x)}{h}$

3 Let $y = f(x)$ be a function. The different notations for derivatives are $y'(x)$ *or* $f'(x)$ or $\dfrac{dy}{dx}$ *or* $\dfrac{d}{dx}f(x)$

DEFINITION

A function f is differentiable at a if $f'(a)$ exists.

A function f differentiable on an open interval $(a,b)[or\ (a,\infty)\ or\ (-\infty,a)\ or\ (-\infty,\infty)]$ if it is differentiable at every interior number in the interval.

A function f differentiable on an closed interval $[a,b]$ if it is differentiable at every interior and end numbers in the interval.

THEOREM : If f is differentiable at a, then f is continuous at a.

NOTE :The converse of Theorem is false; that is, there are functions that are continuous but not differentiable.

Interpretations of Derivative

1 The differential coefficient $\frac{dy}{dx}$ of the function $y = f(x)$ at any point of the curve represents slope or gradient of the curve at that point (equivalently the slope of the tangent at that point).

2 If $f(x)$ represents a quantity at x, then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x = a$.

Note

If $\frac{dy}{dx} = 0,$ then the tangent is parallel to x-axis.

If $\frac{dy}{dx} = \infty,$ then the tangent is perpendicular to x-axis.

Problems on derivative by definition.

1. Show that the function $f(x) = x^2$ is derivable on [0, 1]

Let $'a'$ be any interior point of the interval $[0,1]$. Then

$$f'(a) = \underset{x\to a}{Lt}\frac{f(x)-f(a)}{x-a} = \underset{x\to a}{Lt}\frac{x^2-a^2}{x-a} = \underset{x\to a}{Lt}\frac{(x-a)(x+a)}{x-a} = a+a = 2a,\ exists\ \ \&\ finite$$

$$f'(0) = \underset{x\to 0^+}{Lt}\frac{f(x)-f(0)}{x-0} = \underset{x\to 0^+}{Lt}\frac{x^2-0}{x-0} = \underset{x\to 0^+}{Lt}\ x = 0,\ exists\ \&\ finite$$

$$f'(1) = \underset{x\to 1^-}{Lt}\frac{f(x)-f(1)}{x-1} = \underset{x\to 1^-}{Lt}\frac{x^2-1}{x-1} = \underset{x\to 1^-}{Lt}\frac{(x-1)(x+1)}{x-1} = 1+1 = 2,\ exists\ \&\ finite$$

Since f' exists at all points of the interval, $f(x) = x^2$ is derivable in [0,1].

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2. Find the derivative of $f(x) = \sin x$.

By definition,

$$f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\sin(x+h) - \sin x}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \sin x \underset{h \to 0}{Lt} \frac{\cos h - 1}{h} + \cos x \underset{h \to 0}{Lt} \frac{\sin h}{h}$$

$$= \sin x.(0) + \cos x.(1)$$

$$= \cos x$$

3. Find the derivative of $f(x) = \cos x$.

By definition,

$$f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\cos(x+h) - \cos x}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \cos x \underset{h \to 0}{Lt} \frac{\cos h - 1}{h} - \sin x \underset{h \to 0}{Lt} \frac{\sin h}{h}$$

$$= \cos x.(0) - \sin x.(1)$$

$$= -\sin x$$

.

4. Let $f(x) = \frac{1}{4}x^2 + 1$. Find $f'(-1)$ and $f'(-3)$.

We know that

$$f'(-1) = \underset{x \to -1}{Lt} \frac{f(x) - f(-1)}{x - (-1)}$$

$$= \underset{x \to -1}{Lt} \frac{\frac{1}{4}x^2 + 1 - \frac{5}{4}}{x - (-1)}$$

$$= \underset{x \to -1}{Lt} \frac{\frac{1}{4}x^2 - \frac{1}{4}}{x + 1}$$

$$= \underset{x \to -1}{Lt} \frac{\frac{1}{4}\left(x^2 - 1\right)}{x + 1}$$

$$= \underset{x \to -1}{Lt} \frac{\frac{1}{4}(x-1)(x+1)}{x + 1}$$

We know that

$$f'(3) = \underset{x \to 3}{Lt} \frac{f(x) - f(3)}{x - (3)}$$

$$= \underset{x \to 3}{Lt} \frac{\frac{1}{4}x^2 + 1 - \frac{13}{4}}{x - 3}$$

$$= \underset{x \to 3}{Lt} \frac{\frac{1}{4}x^2 - \frac{9}{4}}{x - 3}$$

$$= \underset{x \to 3}{Lt} \frac{\frac{1}{4}\left(x^2 - 9\right)}{x - 3}$$

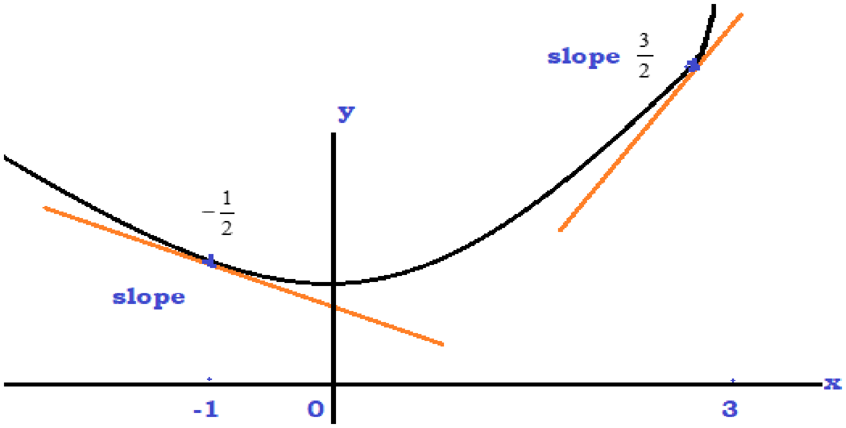
$$= \underset{x \to 3}{Lt} \frac{\frac{1}{4}(x-3)(x+3)}{x - 3}$$

$$= \underset{x \to -1}{Lt} \frac{1}{4}(x-1)$$

$$= \frac{1}{4}(-1-1) = -\frac{1}{2}$$

$$= \underset{x \to 3}{Lt} \frac{1}{4}(x+3)$$

$$= \frac{1}{4}(3+3) = \frac{3}{2}$$



Results

The graph of f rises from left to right if $f'(a) > 0$

If $|f'(a)|$ is large, then the graph of f is very steep near the point

.

The graph of f falls from left to right if $f'(a) < 0$

If $|f'(a)|$ is small, then the graph of f is nearly horizontal near the point

5. Let $f(x) = x^2$. Show that $f'(x) = 2x$ for all x.

$$f'(x) = \underset{t \to x}{Lt} \frac{f(t)-f(x)}{t-x}$$

$$= \underset{t \to x}{Lt} \frac{t^2-x^2}{t-x}$$

$$= \underset{t \to x}{Lt} \frac{(t-x)(t+x)}{t-x}$$

$$= \underset{t \to x}{Lt} (t+x)$$

$$= x + x = 2x$$

6. Let $f(x) = x^{\frac{1}{2}}$. Show that $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ for $x > 0$**.**

$$f'(x) = \underset{t \to x}{Lt} \frac{f(t)-f(x)}{t-x}$$

$$= \underset{t \to x}{Lt} \frac{t^{\frac{1}{2}} - x^{\frac{1}{2}}}{t-x}$$

$$= \underset{t \to x}{Lt} \frac{t^{\frac{1}{2}} - x^{\frac{1}{2}}}{t-x} \cdot \frac{t^{\frac{1}{2}} + x^{\frac{1}{2}}}{t^{\frac{1}{2}} + x^{\frac{1}{2}}}$$

$$= \underset{t \to x}{Lt} \frac{t-x}{t-x} \cdot \frac{1}{t^{\frac{1}{2}} + x^{\frac{1}{2}}} \qquad \because (a-b)(a+b) = a^2 - b^2$$

$$= \underset{t \to x}{Lt} \frac{1}{t^{\frac{1}{2}} + x^{\frac{1}{2}}}$$

$$= \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{2}}} = \frac{1}{2}\frac{1}{x^{\frac{1}{2}}} = \frac{1}{2}x^{-\frac{1}{2}}$$

.

7. Find the derivative of the function $f(x)=x^2-4x+5$ at a point a.

Let $f(x)=x^2-4x+5$

$$f'(a)=\underset{h\to 0}{Lt}\frac{f(a+h)-f(a)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{(a+h)^2-4(a+h)+5-(a^2-4a+5)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{a^2+h^2+2ah-4a-4h+5-a^2+4a-5}{h}$$

$$=\underset{h\to 0}{Lt}\frac{h^2+2ah-4h}{h}$$

$$=\underset{h\to 0}{Lt}\frac{h(h+2a-4)}{h}$$

$$=2a-4$$

8. Find the derivative of the function $f(x)=x^2-4x+5$ using the definition of the derivative.

Let $f(x)=x^2-4x+5$

$$f'(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{\left((x+h)^2-4(x+h)+5\right)-\left(x^2-4x+5\right)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{\left(x^2+h^2+2xh-4x-4h+5\right)-\left(x^2-4x+5\right)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{\left(h^2+2xh-4h\right)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{h\left(h+2x-4\right)}{h}$$

$$=2x-4$$

9. Where is the function $f(x)=|x|$ differentiable?

If $x>0$, then $|x|=x$ and hence $|x+h|=x+h$.

Therefore, for $x>0$, we have

$$f'(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{|x+h|-|x|}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)-x}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}$$

If $x<0$, we have $|x|=-x$ and hence $|x+h|=-(x+h)$.

Therefore, for $x<0$,

$$f'(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{|x+h|-|x|}{h}$$

$$=\lim_{h\to 0}\frac{-(x+h)-(-x)}{h}$$

$$=\lim_{h\to 0}\frac{-h}{h}=\lim_{h\to 0}(-1)=-1$$

$= 1$

and so f is differentiable for any $x > 0$.

and so f is differentiable for any $x < 0$.

For $x = 0$ we have to investigate

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{|0+h|-|0|}{h}$$

$$= \lim_{h\to 0}\frac{|h|}{h}$$

Let's compute the left and right limits separately: We know that $|h| = \begin{matrix} h, & x>0 \\ -h, & x<0 \end{matrix}$

$$\lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

and

$$\lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0^-}\frac{-h}{h} = \lim_{h\to 0^-}(-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at (0,0) .

10. Find by the definition of derivative, $f'(x)$ if $f(x) = \sqrt{x}$.

We know that

$$f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \underset{h \to 0}{Lt} \frac{x+h-x}{h\ \sqrt{x+h} - \sqrt{x}}$$

$$= \underset{h \to 0}{Lt} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Here $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$.

11. **Find f' if $f(x) = \frac{1-x}{2+x}$ by definition of derivative.**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h}$$

$$= \lim_{h \to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{(2-x-2h-x^2-xh) - (2-x+h-x^2-xh)}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-3h}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-3}{(2+x+h)(2+x)}$$

$$= -\frac{3}{(2+x)^2} \ .$$

12. **Determine whether $f'(0)$ exists or not for** $f(x)=\left|\begin{array}{ll} x\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{array}\right.$

We know that

$$f'(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$f'(0)=\underset{h\to 0}{Lt}\frac{f(h)-f(0)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{h\sin\frac{1}{h}-0}{h}$$

$$=\underset{h\to 0}{Lt}\sin\frac{1}{h}$$

This limit does not exist because it takes the values -1 *and* 1 in any interval containing 0. Therefore $f'(0)$ does not exist.

13. **Determine whether $f'(0)$ exists or not for** $f(x)=\left|\begin{array}{ll} x^2\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{array}\right.$

We know that

$$f'(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$f'(0)=\underset{h\to 0}{Lt}\frac{f(h)-f(0)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{h^2\sin\frac{1}{h}-0}{h}$$

$$=\underset{h\to 0}{Lt}h\sin\frac{1}{h}$$

We know that $-1\leq\sin\frac{1}{h}\leq 1$

Therefore $-|h|\leq|h|\sin\frac{1}{h}\leq|h|$

i.e. $-|h|\leq h\sin\frac{1}{h}\leq|h|$

Also $\underset{h\to 0}{Lt}-|h|=0$ *and* $\underset{h\to 0}{Lt}|h|=0$.

Therefore by Squeeze theorem,

$$\underset{h\to 0}{Lt}h\sin\frac{1}{h}=0=f(0)$$

Therefore $f'(0)$ exists and $f'(0)=0$.

14. Find the derivative of $f(x) = e^x$.

By definition,

$$f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{e^{x+h} - e^x}{h}$$

$$= e^x \underset{h \to 0}{Lt} \frac{e^h - 1}{h}$$

$$= e^x.1 = e^x$$

To evaluate $\underset{h \to 0}{Lt} \frac{e^h - 1}{h}$

Let $y = e^h - 1$. Then $y + 1 = e^h$ *i.e.* $h = \log(1+y)$

Also, when $h \to 0$, $y \to 0$

$$\underset{h \to 0}{Lt} \frac{e^h - 1}{h} = \underset{y \to 0}{Lt} \frac{y}{\log(1+y)}$$

$$= \underset{y \to 0}{Lt} \frac{1}{\log(1+y)^{\frac{1}{y}}}$$

$$= \frac{1}{\log e} = 1$$

14. Find the derivative of $f(x) = x^n$, where n is a rational number.

Case 1: When n is positive integer.

By definition,

$$f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{(x+h)^n - x^n}{h}$$

$$= \underset{h \to 0}{Lt} \frac{1}{h}\left[x^n + \frac{n}{1!}x^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \ldots\ldots + h^n - x^n\right]$$

$$= \underset{h \to 0}{Lt} \left[\frac{n}{1!}x^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \ldots\ldots + h^{n-1}\right]$$

$$= n\ x^{n-1}$$

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Case 2: When $n = \frac{p}{q}$ where p, q are positive integers.

Let $x = y^q \ \ and \ \ a = b^q$. Then as $x \to a, \ y^q \to b^q$.
So $x \to a \ \ means \ \ y \to b.$

Consider

$$\frac{f(x) - f(a)}{x-a} = \frac{x^n - a^n}{x-a}$$

$$= \frac{x^{p/q} - a^{p/q}}{x-a}$$

$$= \frac{\left(y^q\right)^{p/q} - \left(b^q\right)^{p/q}}{y^q - b^q}$$

$$= \frac{y^p - b^p}{y^q - b^q}$$

Taking limit on both sides

$$\underset{x \to a}{Lt} \frac{f(x) - f(a)}{x-a} = \underset{x \to a}{Lt} \frac{y^p - b^p}{y^q - b^q}$$

$$= \frac{\underset{x \to a}{Lt} \frac{y^p - b^p}{y-b}}{\underset{x \to a}{Lt} \frac{y^q - b^q}{y-b}}$$

$$= \frac{pb^{p-1}}{qb^{q-1}}$$

$$= \frac{p}{q}.\left(b^q\right)^{\frac{p}{q}-1}$$

$$f'(a) = n.a^{n-1}$$

Case 3: When $n = -m$, where m is positive integer or fraction.

Consider

$$\frac{f(x) - f(a)}{x-a} = \frac{x^n - a^n}{x-a}$$

$$= \frac{x^{-m} - a^{-m}}{x-a}$$

$$= \frac{a^m - x^m}{x^m a^m (x-a)}$$

$$= -\frac{x^m - a^m}{x^m a^m (x-a)}$$

Taking limit on both sides

$$\underset{x \to a}{Lt} \frac{f(x) - f(a)}{x-a} = -\underset{x \to a}{Lt} \frac{x^m - a^m}{x^m a^m (x-a)}$$

$$= -\frac{1}{a^m} \underset{x \to a}{Lt} \frac{1}{x^m} \underset{x \to a}{Lt} \frac{x^m - a^m}{x-a}$$

$$= -\frac{1}{a^m}.\frac{1}{a^m}.ma^{m-1}$$

$$= -ma^{-m-1}$$

$$f'(a) = n.a^{n-1}$$

Therefore $f'(x) = nx^{n-1}$ for all rational values of n.

HIGHER DERIVATIVES

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f"$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f. Using Leibnitz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}$$

1. If $f(x) = x^3 - x$, find $f'(x)$ and $f''(x)$ and interpret $f''(x)$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[(x+h)^3-(x+h)]-[x^3-x]}{h}$$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-h}{h}$$

$$= \lim_{h\to 0}(3x^2+3xh+h^2-1)$$

$$= 3x^2-1$$

$$f"(x) = \underset{h\to 0}{Lt}\frac{f'(x+h)-f'(x)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{\left[3(x+h)^2-1\right]-\left(3x^2-1\right)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{3x^2+3h^2+6xh-1-3x^2+1}{h}$$

$$= \underset{h\to 0}{Lt}\frac{3h^2+6xh}{h}$$

$$= \underset{h\to 0}{Lt}\,3h+6x$$

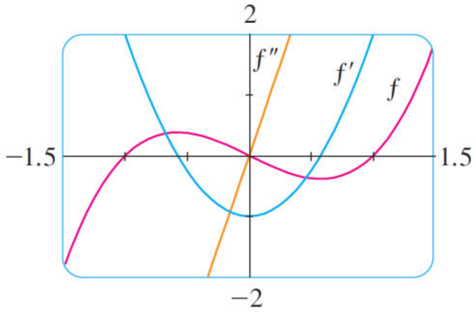
$$= 6x$$

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$.

In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope.

So the graphs serve as a check on our calculations.



Standard Formulas for differentiation

$$\left(cons\tan t\right)' = 0$$

$$\left(x^{n}\right)' = nx^{n-1}$$

$$\left((ax+b)^{n}\right)' = a.n(ax+b)^{n-1}$$

$$\left(e^{ax}\right)' = a.e^{ax}$$

$$\left(a^{x}\right)' = a^{x}\ln a$$

$$\left(\sin ax\right)' = a.\cos ax$$

$$\left(\cos ax\right)' = -a.\sin ax$$

$$\left(\tan ax\right)' = a.\sec^{2} ax$$

$$\left(\cot ax\right)' = -a.\cos ec^{2} ax$$

$$\left(\sec ax\right)' = a.\sec ax.\tan ax$$

$$\left(\cos ecax\right)' = -a.co\sec ax.\cot ax$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^{2}}$$

$$\left(\log_{a} x\right)' = \frac{\log_{a} e}{x}$$

$$\left(\ln x\right)' = \frac{1}{x}$$

$$\left(\sin^{-1} x\right)' = \frac{1}{\sqrt{1-x^{2}}}$$

$$\left(\cos^{-1} x\right)' = -\frac{1}{\sqrt{1-x^{2}}}$$

$$\left(\tan^{-1} x\right)' = \frac{1}{1+x^{2}}$$

$$\left(\cot^{-1} x\right)' = -\frac{1}{1+x^{2}}$$

$$\left(\sinh x\right)' = \cosh x$$

$$\left(\cosh x\right)' = \sinh x$$

RULES OF DIFFERENTIATION

If c is a constant and f, g are differentiable function, then

$$\left(cu\right)' = c.u'$$

$$\left(u \pm v\right)' = u' + v'$$

$$\left(uv\right)' = u.v' + v.u'$$

$$\left(\frac{u}{v}\right)' = \frac{v.u' + u.v'}{v^{2}}$$

$$\frac{du}{dx} = \frac{du}{dy}.\frac{dy}{dx}$$

Addition rules of differentiation

If f and g are differentiable, then $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

1. **Find the derivative of** $y = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$

$$\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$$

$$= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$

$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) — 6(1) + 0$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

2. **Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is Horizontal tangents occur where the derivative is zero.**

We have $\frac{dy}{dx} = \frac{d}{dx}(x^4) - 6\frac{d}{dx}(x^2) + \frac{d}{dx}(4)$

$$= 4x^3 - 12x + 0$$

$$= 4x(x^2 - 3)$$

Thus $dy/dx = 0$ if $x = 0$ or $x^2 - 3 = 0$, that is, $x = \pm\sqrt{3}$.

So the given curve has horizontal tangents at $x = 0, \sqrt{3}$, and $-\sqrt{3}$.

The corresponding points are $(0,4)$, $(\sqrt{3}, -5)$, and $(-\sqrt{3}, -5)$.

3. **The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?**

The velocity and acceleration are

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3$$

$$a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration after 2 s is $a(2) = 14cm/s^2$.

PRODUCT RULES OF DIFFERENTIATION

If f and g are differentiable, then $\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

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1. (a) If $f(x) = xe^x$, find $f'(x)$. (b) Find the nth derivative, $f^{(n)}(x)$.

(a) By the Product Rule, we have

$$f'(x) = \frac{d}{dx}(xe^x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$= xe^x + e^x \cdot 1$$

$$= (x+1)e^{\mathrm{x}}$$

(b) Using the Product Rule a second time, we get

$$f''(x) = \frac{d}{dx}[(x+1)e^x] = (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1)$$

$$= (x+1)e^x + e^x \cdot 1$$

$$= (x+2)e^{\mathrm{x}}$$

Further applications of the Product Rule give

$$f'''(x) = (x+3)e^x$$

$$f^{(4)}(x) = (x+4)e^x$$

In fact, each successive differentiation adds another term e^x, so

$$f^{(n)}(x) = (x+n)e^x ..$$

2. Differentiate the function $f(t) = \sqrt{t}(a+bt)$.

Using the Product Rule, we have

$$f'(t) = \sqrt{t}\frac{d}{dt}(a+bt) + (a+bt)\frac{d}{dt}(\sqrt{t})$$

$$= \sqrt{t} \cdot b + (a+bt) \cdot \frac{1}{2}t^{-1/2}$$

$$= b\sqrt{t} + \frac{a+bt}{2\sqrt{t}}$$

$$= \frac{a+3bt}{2\sqrt{t}}$$

Using the addition Rule. $f(t) = a\sqrt{t} + bt\sqrt{t}$

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$$= at^{1/2} + bt^{3/2}$$

$$f'(t) = \tfrac{1}{2}\, at^{-1/2} + \tfrac{3}{2} bt^{1/2}$$

3. Find $\frac{dy}{dx}$ **if** $y = x^2 e^{2x}\left(x^2+1\right)^4$**.**

$$y = x^2 e^{2x}\left(x^2+1\right)^4$$

$$\frac{dy}{dx} = x^2 e^{2x} 4\left(x^2+1\right)^3 2x + x^2 2e^{2x}\left(x^2+1\right)^4 + 2xe^{2x}\left(x^2+1\right)^4$$

$$\frac{dy}{dx} = e^{2x}\left(x^2+1\right)^3\left[8x^3 + 2x^2\left(x^2+1\right) + 2x\left(x^2+1\right)\right]$$

$$\frac{dy}{dx} = e^{2x}\left(x^2+1\right)^3\left[8x^3 + 2x^4 + 2x^2 + 2x^3 + 2x\right]$$

$$\frac{dy}{dx} = e^{2x}\left(x^2+1\right)^3\left[2x^4 + 10x^3 + 2x^2 + 2x\right]$$

4. Differentiate $y = x^2 \sin x$.

Using the Product Rule, we have

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$

$$= x^2 \cos x + 2x \sin x$$

5. Differentiate $y = (2x+1)^5 (x^3 - x + 1)^4$**.**

Using the Product Rule, we have

$$\frac{dy}{dx} = (2x+1)^5 \frac{d}{dx}(x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx}(2x+1)^5$$

$$\frac{dy}{dx} = \left(2x+1\right)^5 4\left(x^3 - x + 1\right)\left(3x^2 - 1\right) + \left(x^3 - x + 1\right)^4 5\left(2x+1\right)\left(2\right)$$

$$\frac{dy}{dx} = 4\left(2x+1\right)^5\left(x^3 - x + 1\right)\left(3x^2 - 1\right) + 10\left(x^3 - x + 1\right)^4\left(2x+1\right)$$

DIVISION RULES OF DIFFERENTIATION

If f and g are differentiable, then $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$

1 Find (1) $\frac{d}{dx}\left(3x^5 \log x\right)$ **and (2)** $\frac{d}{dx}\left(\frac{x^3}{3x-2}\right)$

(1) $\frac{d}{dx}\left(3x^5 \log x\right) = 3\left[x^5.\frac{1}{x}+5x^4 \log x\right] = 3x^4\left[1+5\log x\right]$

(2) $\frac{d}{dx}\left(\frac{x^3}{3x-2}\right) = \frac{3(3x-2)x^2-3x^3}{(3x-2)^2} = \frac{6x^3-2x^2}{(3x-2)^2}$

2 Find the derivative of $f(x) = ta\,\mathrm{n}\,x$ **.**

$$\frac{d}{dx}ta\,\mathrm{n}\,x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$= \frac{(\cos x)\frac{d}{dx}\sin x - \sin x\frac{d}{dx}\cos x}{(\cos x)^2}$$

$$= \frac{\cos x.\cos x - \sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{(\cos x)^2}$$

$$= \frac{1}{(\cos x)^2}$$

$$= \sec^2 x$$

3 Find the derivative of $f(x) = \cot x$ **.**

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x}$$

$$= \frac{(\sin x)\frac{d}{dx}\cos x - \cos x\frac{d}{dx}\sin x}{(\sin x)^2}$$

$$= \frac{\sin x.(-\sin x) - \cos x(\cos x)}{(\sin x)^2}$$

$$= \frac{-\left(\cos^2 x + \sin^2 x\right)}{(\sin x)^2}$$

$$= \frac{-1}{(\sin x)^2}$$

$$= -co\sec^2 x$$

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4 If $y = \frac{x^2+x-2}{x^3+6}$, find y'

Let $y = \frac{x^2+x-2}{x^3+6}$. Then

$$y' = \frac{(x^3+6)\frac{d}{dx}(x^2+x-2) - (x^2+x-2)\frac{d}{dx}(x^3+6)}{(x^3+6)^2}$$

$$= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{(2x^4+x^3+12x+6) - (3x^4+3x^3-6x^2)}{(x^3+6)^2}$$

$$= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}$$

5 Find the differential coefficients of $\frac{(a-x)^2(b-x)^3}{(c-2x)^3}$

Let $y = \frac{(a-x)^2(b-x)^3}{(c-2x)^3}$

$$\frac{dy}{dx} = \frac{(c-2x)^3\left[-3(a-x)^2(b-x)^2 - 2(b-x)^3(a-x)\right] + 6(a-x)^2(b-x)^3(c-2x)^2}{(c-2x)^6}$$

$$\frac{dy}{dx} = \frac{\left[-3(a-x)^2(b-x)^2(c-2x) - 2(b-x)^3(a-x)(c-2x)\right] + 6(a-x)^2(b-x)^3}{(c-2x)^4}$$

$$\frac{dy}{dx} = \frac{(a-x)(b-x)^2\left\{-3(a-x)(c-2x) - 2(b-x)(c-2x) + 6(a-x)(b-x)\right\}}{(c-2x)^4}$$

Problems on Application of Differentiation

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1 If $x^2+y^2=25$, then find $\frac{dy}{dx}$ and also find an equation of the tangent line to the curve $x^2+y^2=25$ at the point (3,4).

Given $x^2+y^2=25$

Differentiate w.r.t. x,

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

$$\frac{dy}{dx}\ at\ (3,4)\ is\ m=-\frac{3}{4}$$

The equation of tangent at (x_1,y_1) is

$$y-y_1=m(x-x_1)$$

Therefore the equation of tangent at $(3,4)$ is

$$y-4=-\frac{3}{4}(x-3)$$

$$4y-16=-3x+9$$

$$4y+3x=25$$

2 Find equations of the tangent line and normal line to the curve $y=x\sqrt{x}$ at the point $(1,1)$.

The derivative of $f(x)=x\sqrt{x}=xx^{1/2}=x^{3/2}$ is

$$f'(x)=\frac{3}{2}x^{(3/2)-1}=\frac{3}{2}x^{1/2}=\frac{3}{2}\sqrt{x}$$

So, the slope of the tangent line at (1, 1) is $f'(1)=\frac{3}{2}$.

Therefore, an equation of the tangent line is

$$y-1=\frac{3}{2}(x-1)\text{ or }y=\frac{3}{2}x-\frac{1}{2}$$

The normal line is perpendicular to the tangent line, so its slope is the negative reciprocal of $\frac{3}{2}$, that is, $\frac{2}{3}$. Thus an equation of the normal line is

$$y-1=-\frac{2}{3}(x-1)\text{ or }y=-\frac{2}{3}x+\frac{5}{3}$$

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3 Find the points on the curve $y=x^4-6x^2+4$ where the tangent line is horizontal.

If tangent is horizontal, then slope is 0.

Given $y=x^4-6x^2+4$

Then $\frac{dy}{dx}=4x^3-12x$

$$\frac{dy}{dx}=0\ \Rightarrow\ 4x^3-12x=0$$

$$i.e.\ 4x(x^2-3)=0$$

$$i.e.\ x=0,\ x^2-3=0$$

$$i.e.\ x=0,\ x=\pm\sqrt{3}$$

When $x=0,\ y=4$

When $x=\pm\sqrt{3},\ y=9-6\times3+4=-5$

4 The equation of motion of a particle is $s=2t^3-5t^2+3t+4$ where s is measured in cm and time t in seconds. Find the acceleration as function of time. Also what is the acceleration after 2 seconds.

Given $s=2t^3-5t^2+3t+4$

Velocity and acceleration

$$v(t)=\frac{ds}{dt}=6t^2-10t+3$$

$$a(t)=\frac{dv}{dt}=12t-10$$

$$Acceleration\ a(2)=24-10=14cm/\sec^2$$

Therefore the points where the tangents horizontal are $(0,4), \left(\sqrt{3},-5\right), \left(-\sqrt{3},-5\right)$

5. **Find an equation of the tangent line to the curve $y=\dfrac{e^x}{1+x^2}$ at the point $\left(1,\dfrac{e}{2}\right)$.**

Given $y=\dfrac{e^x}{1+x^2}$

$$\frac{dy}{dx}=\frac{(1+x^2)\frac{d}{dx}(e^x)-e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$=\frac{(1+x^2)e^x-e^{\mathrm{x}}(2x)}{(1+x^2)^2}$$

$$=\frac{e^x(1-x)^2}{(1+x^2)^2}$$

So the slope of the tangent line at $\left(1,\dfrac{e}{2}\right)$ is $m=\left[\dfrac{dy}{dx}\right]_{x=1}=0$

$$\frac{dy}{dx}=0\ at\ x=1$$

This means that the tangent line at $\left(1,\dfrac{e}{2}\right)$ is horizontal and its equation is $y-\dfrac{e}{2}=0(x-1)$.

i.e. $y=\dfrac{e}{2}$

6. **Differentiate $f(x)=\frac{\sec x}{1+\tan x}$. For what values of x does the graph of f have a horizontal tangent?**

The Quotient Rule gives

$$f'(x)=\frac{(1+\tan x)\frac{d}{dx}(\sec x)-\sec x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$=\frac{(1+\tan x)\sec x\tan x-\sec x\cdot\sec^2 x}{(1+\tan x)^2}$$

$$=\frac{\sec x(\tan x+\tan^2 x-\sec^2 x)}{(1+\tan x)^2}$$

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$$= \frac{\sec x(\tan x + \sec^2 x - 1 - \sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1 + \tan x)^2}$$

If tangent is horizontal, then slope is 0. Since $\sec x$ is never 0, we see that $f'(x) = 0$ when

$\tan x = 1$, and this occurs when $x = n\pi + \pi/4$, where n is an integer.

7. At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

Since $y = e^x$, we have $y' = e^x$.

Let the x-coordinate of the point in question be a. Then the slope of the tangent line at that point is e^a. This tangent line will be parallel to the line $y = 2x$ if it has the same slope, that is, 2.

Equating slopes, we get

$$e^a = 2$$

$$lne^a = \ln 2$$

$$a = \ln 2$$

Therefore, the required point is $(a, e^a) = (\ln 2, 2)$.

THE CHAIN RULE

If g is differentiable at x and f is differentiable at (x) , then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

THE POWER RULE COMBINED WITH THE CHAIN RULE

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

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Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

The Chain Rule

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

1 Find $\frac{dy}{dx}$ if $y = \sqrt{x^2 - 4x + 7}$

Let $y = \sqrt{u}$ *and* $u = x^2 - 4x + 7$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{2\sqrt{u}}.(2x-4)$$

$$= \frac{1}{2\sqrt{x^2 - 4x + 7}}.(2x-4)$$

2 Find $\frac{dy}{dx}$ if $y = \log \sin \sqrt{x}$

Let $y = \log u,\ u = \sin v$ *and* $v = \sqrt{x}$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx} = \frac{1}{u}.\cos v.\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sin v}.\cos v.\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}.\frac{1}{\sin\sqrt{x}}.\cos\sqrt{x}$$

3. **Differentiate $y = (x^3 - 1)^{100}$.**

Taking $u = g(x) = x^3 - 1$ and $n = 100$ in , we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^3-1)^{100} = 100(x^3-1)^{99}\frac{d}{dx}(x^3-1)$$

$$= 100(x^3-1)^{99} \cdot 3x^2 = 300x^2(x^3-1)^{99}$$

4. **Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$.**

First rewrite f as $f(x) = (x^2+x+1)^{-1/3}$

Thus

$$f'(x) = -\frac{1}{3}(x^2+x+1)^{-4/3}\frac{d}{dx}(x^2+x+1)$$

$$= -\frac{1}{3}(x^2+x+1)^{-4/3}(2x+1)$$

5. **Differentiate $y = (2x+1)^5(x^3-x+1)^4$.**

In this example we must use the Product Rule before using the Chain Rule:

$$\frac{dy}{dx} = (2x+1)^5 \frac{d}{dx}(x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx}(2x+1)^5$$

$$= (2x+1)^5 \cdot 4(x^3 - x + 1)^3 \frac{d}{dx}(x^3 - x + 1)$$

$$+(x^3 - x + 1)^4 \cdot 5(2x+1)^4 \frac{d}{dx}(2x+1)$$

$$= 4(2x+1)^5(x^3 - x + 1)^3(3x^2 - 1) + 5(x^3 - x + 1)^4(2x+1)^4 \cdot 2$$

$$\frac{dy}{dx} = 2(2x+1)^4(x^3 - x + 1)^3(17x^3 + 6x^2 - 9x + 3)$$

6. **Differentiate $y = e^{\sec 3\theta}$.**

The outer function is the exponential function, the middle function is the secant function and the inner function is the tripling function. So we have

$$\frac{dy}{d\theta} = e^{\sec 3\theta} \frac{d}{d\theta}(\sec 3\theta)$$

$$= e^{\sec 3\theta} \sec 3\theta \tan 3\theta \frac{d}{d\theta}(3\theta)$$

$$= 3e^{\sec 3\theta} \sec 3\theta \tan 3\theta$$

IMPLICIT DIFFERENTIATION

This method consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'. It is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

Consider the implicit function $F(x,y)=0$. Differentiate F w.r.t x and rearrange the terms as $G(x,y)\frac{dy}{dx}+H(x,y)=0$. Then $\frac{dy}{dx}=-\frac{H(x,y)}{G(x,y)}$.

1. **Find $\frac{dy}{dx}$ for** $xy=1$

Method1:

Given $xy=1$ i.e. $y=\frac{1}{x}$

Differentiate w.r.t. x

Therefore $\frac{dy}{dx}=-\frac{1}{x^2}$

Method 2: Implicit Differentiation

Given $xy=1$

Differentiate w.r.t. x

$x\frac{dy}{dx}+y=0$

$\frac{dy}{dx}=-\frac{y}{x}=-\frac{1/x}{x}=-\frac{1}{x^2}$

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2. Find $\frac{dy}{dx}$ if $ax^2+by^2+2hxy+2gx+2fy+c=0$

Differentiate w.r.t. x, we have

$$a2x+b2y\frac{dy}{dx}+2h\left[x\frac{dy}{dx}+y\right]+2g+2f\frac{dy}{dx}+c=0$$

$$\frac{dy}{dx}\left[2by+2hx+2f\right]=-2ax-2hy-2g-c$$

$$\frac{dy}{dx}=\frac{-2ax-2hy-2g-c}{2by+2hx+2f}$$

3 Find $\frac{dy}{dx}$ if $x^3+y^3=6xy$.

Also find the equation of tangent at (3,3). At what point in the first quadrant is the tangent horizontal?.

Given $x^3+y^3=6xy$

Differentiate w.r.t. x, we have

$$3x^2+3y^2\frac{dy}{dx}=6\left[x\frac{dy}{dx}+y\right]$$

$$3y^2\frac{dy}{dx}-6x\frac{dy}{dx}=6y-3x^2$$

$$\left(3y^2-6x\right)\frac{dy}{dx}=\left(6y-3x^2\right)$$

$$\frac{dy}{dx}=\frac{\left(6y-3x^2\right)}{\left(3y^2-6x\right)}$$

$$m=\left[\frac{dy}{dx}\right]_{(3,3)}=\frac{(18-27)}{(27-18)}=-1$$

Equation of tangent at (3,3) is

$$(y-3)=m(x-3)$$
$$y-3=-1(x-3)$$
$$y-3=-x+3$$
$$y=-x+6$$

Equation tangent is horizontal if $\frac{dy}{dx}=0$.

$$\frac{\left(6y-3x^2\right)}{\left(3y^2-6x\right)}=0$$

$$6y-3x^2=0$$

$$y=\frac{x^2}{2}$$

From the given equation

$$x^3+\frac{x^6}{8}=6x\frac{x^2}{2}$$

$$\frac{x^6}{8}=2x^3$$

$$x^3=16=4^2$$
$$x=4^{\frac{2}{3}}=2^{\frac{4}{3}}$$

$$\therefore\ y=\frac{x^2}{2}=\frac{2^{\frac{8}{3}}}{2}=2^{\frac{5}{3}}$$

$\therefore$The tangent is horizontal at $\left(2^{\frac{4}{3}},2^{\frac{5}{3}}\right)$.

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4 Show that the sum of x *and* y intercepts of any tangent line to the curve $\sqrt{x}+\sqrt{y}=\sqrt{c}$ is equal to c.

Given $\sqrt{x}+\sqrt{y}=\sqrt{c}$

Differentiate w.r.t. x, we have

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

Let (a,b) be a point on the curve. The slope of the tangent at the point is $m=\frac{dy}{dx}=-\frac{\sqrt{b}}{\sqrt{a}}$

$\therefore$ Equation of tangent is

$(y-b)=m(x-a)$

$$(y-b)=-\frac{\sqrt{b}}{\sqrt{a}}(x-a)$$

$$y\sqrt{a}-b\sqrt{a}=-x\sqrt{b}+a\sqrt{b}$$

$$y\sqrt{a}+x\sqrt{b}=a\sqrt{b}+b\sqrt{a}$$

$$\frac{y\sqrt{a}}{a\sqrt{b}+b\sqrt{a}}+\frac{x\sqrt{b}}{a\sqrt{b}+b\sqrt{a}}=1$$

$$\frac{y}{\sqrt{ab}+b}+\frac{x}{a+\sqrt{ab}}=1$$

$\therefore$ the x *and* y intercepts are $a+\sqrt{ab}$ and $b+\sqrt{ab}$ respectively. Now we have to show that their sum is equal to c .

Since (a,b) lies on the line $\sqrt{x}+\sqrt{y}=\sqrt{c}$, we have $\sqrt{a}+\sqrt{b}=\sqrt{c}$

$$\begin{aligned}a+\sqrt{ab}+b+\sqrt{ab}&=a+b+2\sqrt{ab}\\&=\left(\sqrt{a}+\sqrt{b}\right)^2\\&=\left(\sqrt{c}\right)^2\\&=c\end{aligned}$$

5. (a) If $x^2+y^2=25$, find $\frac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2+y^2=25$ at the point $(3,4)$.

(a) Differentiate both sides of the equation $x^2+y^2=25$:

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$2x+2y\frac{dy}{dx}=0$$

Now we solve this equation for dy/dx:

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$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) At the point $(3, 4)$ we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at $(3, 4)$ is therefore

$$y - 4 = -\tfrac{3}{4}(x - 3) \ or \ 3x + 4y = 25$$

Problems on higher order derivatives by formulas

1. Find y'' if $x^2 + y^4 = 10$.

Differentiate w.r.t. x

$$2x + 4y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{2y^3}$$

Again Differentiate w.r.t. x

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\frac{y^3.1 - x.3y^2\frac{dy}{dx}}{\left(y^3\right)^2}$$

$$= -\frac{1}{2}\frac{y^3.1 - x.3y^2\left(-\frac{x}{2y^3}\right)}{\left(y^3\right)^2}$$

$$= -\frac{1}{2}\ \frac{y^3 + \frac{3x^2}{2y}}{y^6}$$

$$= -\frac{1}{2}\ \frac{2y^4 + 3x^2}{2y^7}$$

$$= -\frac{1}{4}\ \frac{y^4 + 2x^2 + \left(y^4 + x^2\right)}{y^7}$$

$$= -\frac{1}{4}\ \frac{y^4 + 2x^2 + 10}{y^7}$$

2 Find y'' if $x^4 + y^4 = 16$.

Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3y' = 0$$

Solving for y' gives

$$y' = -\frac{x^3}{y^3}$$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x:

$$y'' = \frac{d}{dx}(-\frac{x^3}{y^3}) = -\frac{y^3(d/dx)(x^3) - x^3(d/dx)(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3(3y^2y')}{y^6}$$

If we use the expression y', we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2(-\frac{x^3}{y^3})}{y^6}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^7}$$

$$= -\frac{3x^2(y^4 + x^4)}{y^7}$$

$$= -\frac{3x^2(16)}{y^7} \quad \text{since } x^4 + y^4 = 16.$$

$$y'' = -48\frac{x^2}{y^7}$$

LOGARITHMIC DIFFERENTIATION

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called logarithmic differentiation.

DERIVATIVES OF LOGARITHMIC FUNCTIONS

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad and \quad \frac{d}{dx}\text{logx} = \frac{1}{x}\log_{10} e$$

1. Differentiate $y = x^{\sqrt{x}}$.

Taking log$_e$ on both sides

$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\,\text{In}\,x$

2. Differentiate $y = x^x$.

Taking log$_e$ on both sides

$\ln y = \ln x^x = x\,\text{In}\,x$

3 Find $\frac{dy}{dx}$ if $y = (\sin x)^x$

Take log on both sides

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$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right)$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

$$\frac{y'}{y} = x \cdot \frac{1}{x} + (\ln x).1$$

$$y' = y\left(\frac{1}{x} + \ln x\right)$$

$$= x^x\left(\frac{1}{x} + \ln x\right)$$

$$\log y = \log(\sin x)^x$$

$$= x.\log x$$

Differentiate w.r.t.x

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \log x.1$$

$$\frac{dy}{dx} = y(1+\log x)$$

$$= (\sin x)^x(1+\log x)$$

4. **Differentiate** $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \frac{3}{4}\text{In } x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2+1} - 5 \cdot \frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2})$$

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2})$$

5 **Differentiate the function** $y = \frac{x^4}{(1-2x)\sqrt{x^2-1}}$

Taking loge on both sides of the given function, we get

$$\log_e y = \log_e \frac{x^4}{(1-2x)\sqrt{x^2-1}}$$

$$\log_e y = \log_e x^4 - \log_e(1-2x)\sqrt{x^2-1}\text{,} \quad \text{since } \log\frac{a}{b} = \log a - \log b$$

$$\log_e y = \log_e x^4 - \left[\log_e(1-2x) + \log_e\sqrt{x^2-1}\right], \quad \text{since } \log ab = \log a + \log b$$

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Differentiate w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^4}4x^3-\frac{1}{1-2x}(0-2)-\frac{1}{\sqrt{x^2-1}}.\frac{1}{2\sqrt{x^2-1}}(2x)$$

$$\frac{dy}{dx}=y\left[\frac{4}{x}+\frac{2}{1-2x}-\frac{x}{\left(x^2-1\right)}\right]$$

$$\frac{dy}{dx}=\frac{x^4}{(1-2x)\sqrt{x^2-1}}\left[\frac{4}{x}+\frac{2}{1-2x}-\frac{x}{\left(x^2-1\right)}\right]$$

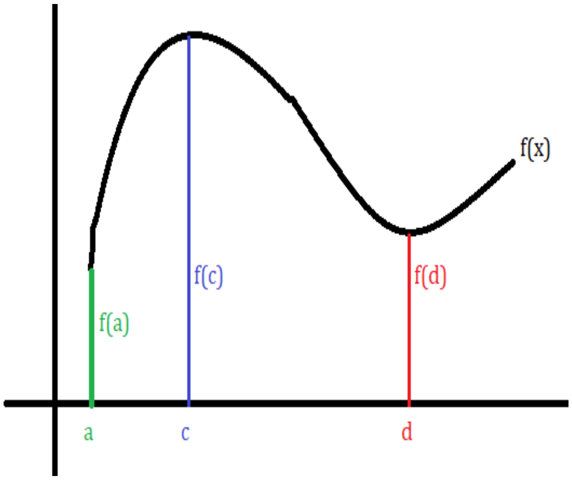
Maximum and Minimum Values

Let c be a number in the domain of the function f. Then $f(c)$ is the

(i) absolute (or global) maximum value of f, if $f(c)\geq f(x), \forall x$

(ii) absolute (or global) minimum value of f, if $f(c)\leq f(x), \forall x$

The maximum or minimum values of f are called extreme values of f.



The above diagram shows the graph of f with absolute maximum at c and absolute minimum at a. Because $[c, f(c)]$ is the highest point and $[a, f(a)]$ is the lowest point.

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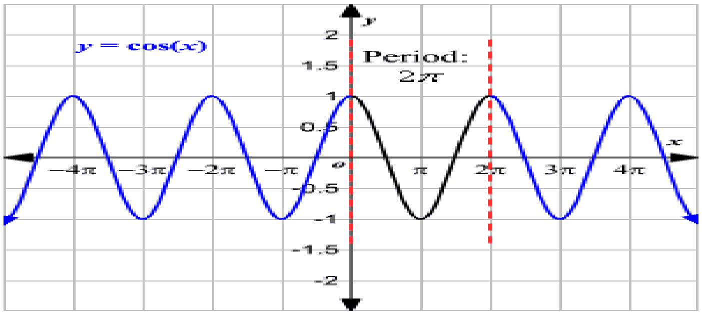
Example:

The function $f(x)=\cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times since $\cos 2n\pi=1, \forall n$ and $-1\leq\cos x\leq 1, \forall x.$ Similarly $\cos(2n+1)\pi=-1, \forall n$ is its minimum value.



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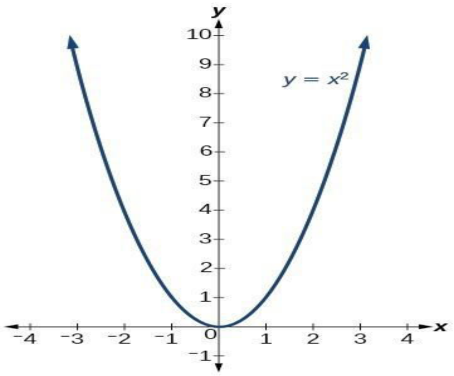
Example:

If $f(x) = x^2$, then $f(x) \geq f(0), \forall x.$ Therefore $f(0) = 0$ is the absolute (and local) minimum of f. This corresponds to the fact that origin is the lowest point on the parabola.

However there is no highest point on the parabola and hence the function has no maximum value.



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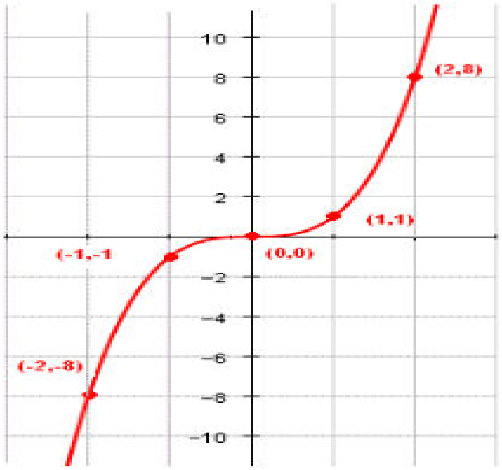
Example:

From the graph of the function $f(x) = x^3$, we observe that the function has neither absolute maximum value nor absolute minimum value.

Also it has no local extreme values.



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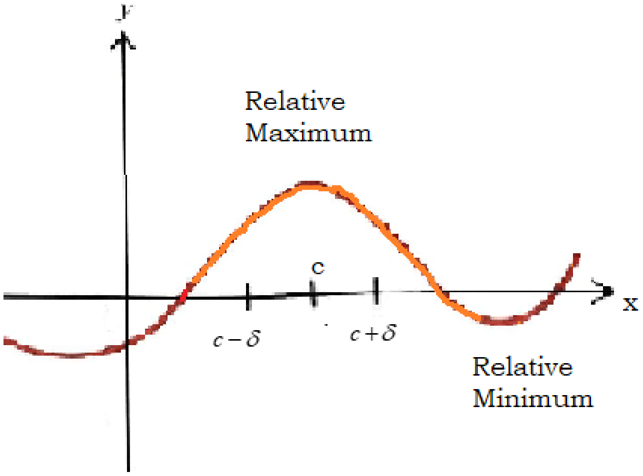
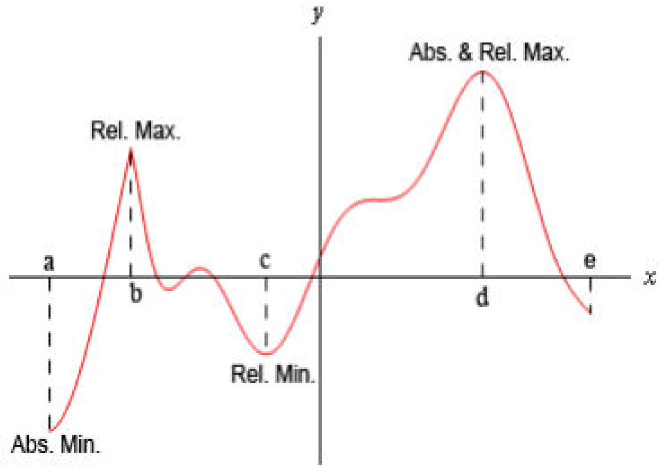
Definition

A function f has a **relative (or local) maximum** value at c, if there is some number $\delta > 0$ such that $f(c)$ is the maximum value in the interval $(c-\delta,\ c+\delta)$.

A function f has a **relative (or local) minimum** value at c, if there is some number $\delta > 0$ such that $f(c)$ is the minimum value in the interval $(c-\delta,\ c+\delta)$.



Definition



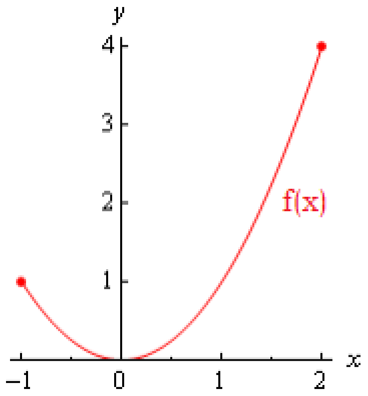
Absolute maximum (or minimum) exists at x = c provided f(c) is the largest (or smallest) value of the function than any other value of f defined in the domain.

Relative maximum (or minimum) exists at x = c provided f(c) is the largest (or smallest) value of the function than any other value of f defined in the interval near x=c.

Absolute extremum may be considered as relative extremum. But converse need not be.

According to definition, relative extremum do not occur at the end points of a domain

Example



Consider the graph of $f(x)$ in the interval $[0,2]$. Identify the extremum of the function.

From the graph it is evident that the absolute maximum exists at $x=2$ and its value is 4.

Also the relative and absolute minimum value exists at $x=0$ and its value is 0.

Relative maximum does not exist for this function.

Theorem: Let f be defined on $[a,b]$. If f has local maximum or minimum at c in (a,b)` and if $f'(x)$ exists then $f'(c)=0$.

Note: A critical number of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ does not exist.

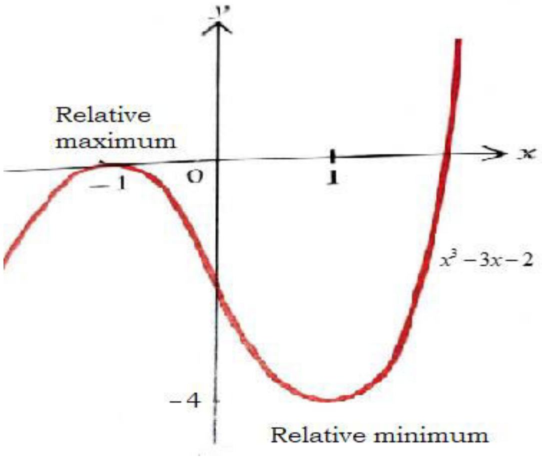
Definition: If f has a local maximum or minimum at c, then c is a critical point of f. But converse need not be true.

Example: Consider the function $f(x)=x^3$.

Here $f'(x) = 3x^2$. $f'(x) = 0$ implies $x^2 = 0$. Therefore the critical point is $x = 0$. At this point no extremum exist.

Example: Find the relative extreme values of

$f(x) = x^3 - 3x - 2$.



$$f'(x) = 3x^2 - 3$$
$$f'(x) = 0 \quad \Rightarrow \quad 3x^2 - 3 = 0, \qquad x = \pm 1$$

We find an interval $(0,2)$ around 1.

Now $f(0) = 0 - 0 - 2 = -2, \quad f(1) = 1 - 3 - 2 = -4,$ and $f(2) = 8 - 6 - 2 = 0$

Here $f(0) \;<\; f(1) \;<\; f(2)$

$\therefore\; f(1)$ is the relative minimum value of f.

Next we find an interval $(-2,0)$ around -1.

Now $f(-2) = -8 + 6 - 2 = -4, \quad f(-1) = -1 + 3 - 2 = 0,$ and $f(0) = 0 - 0 - 2 = -2$

Here $f(-2) \;<\; f(-1) \;<\; f(0)$

$\therefore\; f(-1)$ is the relative maximum value of f.

Example: Find the critical numbers of

$f(x) = x^{\frac{3}{5}}(4 - x)$.

Given

$$f(x) = x^{\frac{3}{5}}(4 - x) = 4x^{\frac{3}{5}} - x^{\frac{8}{5}}$$

$$f'(x) = 4.\frac{3}{5}x^{-\frac{2}{5}} - \frac{8}{5}x^{\frac{3}{5}} = \frac{12}{5}\frac{1}{x^{\frac{2}{5}}} - \frac{8}{5}x^{\frac{3}{5}}$$

Let $f'(x) = 0$

$$\frac{12}{5}\frac{1}{x^{\frac{2}{5}}} - \frac{8}{5}x^{\frac{3}{5}} = 0$$

Multiply by $x^{\frac{2}{5}}$

$$\frac{12}{5} - \frac{8}{5}x^{\frac{5}{5}} = 0$$

$$x = \frac{12}{5}.\frac{5}{8} = \frac{3}{2}$$

Example: Find the critical numbers of

$f(x) = 6x^5 + 33x^4 - 30x^3 + 100$.

Given $f(x) = 6x^5 + 33x^4 - 30x^3 + 100$

$$f'(x) = 30x^4 + 132x^3 - 90x^2$$

Consider $f'(x) = 0$

$$30x^4 + 132x^3 - 90x^2 = 0$$
$$6x^2\left(5x^2 + 22x - 15\right) = 0$$
$$6x^2\left(5x - 3\right)\left(x - 5\right) = 0$$

Therefore the critical points are $x = 0,\; x = 5,\; x = \frac{3}{5}$

Also $f'(x)$ does not exist when $x=0$. Therefore the critical points are $x=0,\frac{3}{2}.$	

THE CLOSED INTERVAL METHOD

To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a,b]$:

I. Find the values of f at the critical numbers of f in (a,b) .

2. Find the values of f at the endpoints of the interval.

3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

1. Let $f(x)=x-x^3,\ \ in\ \ \ [0,1].$ Find the extreme values of f .

$f'(x)=1-3x^2$

$f'(x)=0\ \Rightarrow\ 1-3x^2=0\quad i.e.\ \ x=\pm\frac{1}{\sqrt{3}}$

$\therefore$ Extreme value exists at the end points 0 and 1 and critical point $\frac{1}{\sqrt{3}}$.

The values of f at these points are

$f(0)=0,\quad f(1)=0,\quad f\left(\frac{1}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3}}=\frac{2}{3\sqrt{3}}$

The minimum value of f is 0 and it occurs at 0 and 1.

The maximum value of f is $\frac{2}{3\sqrt{3}}$ and it occurs at $\frac{1}{\sqrt{3}}$.

.

2. Let $f(x)=x-x^3,\ \ in\ \ \ [2,4].$ Find the extreme values of f .

$f'(x)=1-3x^2$

$f'(x)=0\ \Rightarrow\ 1-3x^2=0\quad i.e.\ \ x=\pm\frac{1}{\sqrt{3}}$

But both the critical points lies out of the interval.

$\therefore$ Extreme value exists at the end points 2 and 4 only.

The values of f at these points are

$f(2)=2-8=-6,\quad f(4)=4-64=-60$

The maximum value of f is -6 and it occurs at 2.

The minimum value of f is -60 and it occurs at 4.

3. Let $f(x)=x^3,\ \ in\ \ \ [-2,1].$ Find the extreme values of f .

$f'(x)=3x^2$

$f'(x)=0\ \Rightarrow\ 3x^2=0\quad i.e.\ \ x=0$

$\therefore$ Extreme value exists at the end points –2, 1 and the critical point 0.

The values of f at these points are

$f(-2)=-8,\quad f(1)=1,\quad f(0)=0$

4 Let $f(x)=x^{\frac{2}{3}},\ \ in\ \ \ [-1,1].$ Find the extreme values of f .

$f'(x)=\frac{2}{3}x^{-\frac{1}{3}}=\frac{2}{3}\frac{1}{x^{\frac{1}{3}}}$

$f'(0)$ does not exist.

$\therefore$ Extreme value exists at the end points –1, 1 and the critical point 0.

The values of f at these points are

$f(-1)=1,\quad f(1)=1,\quad f(0)=0$

The maximum value of f is 1 and it occurs at 1.

The minimum value of f is -8 and it occurs at -2.

The maximum value of f is 1 and it occurs at -1 and 1.

The minimum value of f is 0 and it occurs at 0.

5. Find the absolute maximum and absolute minimum values of the function

$3x^4 - 4x^3 - 12x^2 + 1$ **on the interval [–2,3].**

Here f is continuous on [–2,3].

Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

Since $f'(x)$ for all x in the interval, the only critical values are given by $f'(x) = 0$.

$$12x\left(x^2 - x - 12\right) = 0$$
$$x(x-4)(x+3) = 0$$

i.e. $x = 0,\ 4,\ -3$ are the critical values and $x = 0$ only lies in the interval [–2,3].

The value of f at the end points are $f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 1 = 48 + 32 - 48 + 1 = 33$

$f(3) = 3(3)^4 - 4(3)^3 - 12(3)^2 + 1 = 243 - 108 - 108 + 1 = 38$

The value of f at this critical point is $f(0) = 3(0)^4 - 4(0)^3 - 12(0)^2 + 1 = 1$

Comparing these three values, we see that the absolute maximum value is $f(3) = 38$ and the absolute minimum value is $f(0) = 1$

6. Find the absolute maximum and minimum values of the function

$$\boldsymbol{f(x) = x^3 - 3x^2 + 1, \quad \frac{1}{2} \le x \le 4}$$

Since f is continuous on $[\frac{1}{2}, 4\,]$, we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$
$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $(\frac{1}{2}, 4)$.

The values of f at these critical numbers are

$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the endpoints of the interval are $f(\frac{1}{2}) = \frac{1}{8}$ $\qquad f(4) = 17$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

7. Find the absolute maximum and minimum values of $f(x) = x^3 - 3x^2 + 1, -\frac{1}{2} \le x \le 4.$

Given $f(x)$ is continuous in the closed interval $\left[-\frac{1}{2}, 4\right]$.

$f'(x) = 3x^2 - 6x$. Since $f'(x)$ exists for all x, the only critical values are given by $f'(x) = 0$.

$$f'(x) = 0 \Rightarrow 3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

$$\therefore x = 0, 2 \text{ are the critical points.}$$

Now let us find the value of f at the critical and end points.

$$f\left(-\frac{1}{2}\right) = -\frac{1}{8} - 3 \times \frac{1}{4} + 1 = -\frac{1}{8} - \frac{3}{4} + 1 = \frac{1}{8}$$

$$f(0) = 0 - 3(0) + 1 = 1$$

$$f(2) = 8 - 3 \times 4 + 1 = 8 - 12 + 1 = -3$$

$$f(4) = 64 - 3 \times 16 + 1 = 17$$

From the above, the absolute maximum exists at $x = 4$ and its maximum value is 17 and

the absolute minimum exists at $x = 2$ and its minimum value is –3.

8. Find the absolute maximum and minimum values of $f(x) = x - 2\sin x, 0 \le x \le 2\pi.$

Given $f(x)$ is continuous in the closed interval $[0, 2\pi]$.

$f'(x) = 1 - 2\cos x$. Since $f'(x)$ exists for all x, the only critical values are given by $f'(x) = 0$.

$$f'(x) = 0 \Rightarrow 1 - 2\cos x = 0$$

$$i.e. \quad \cos x = \frac{1}{2}$$

$$\therefore x = \frac{\pi}{3}, \frac{5\pi}{3} \text{ are the critical points.}$$

Now let us find the value of f at the critical and end points.

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$$f(0) = 0 - 2\sin 0 = 0$$

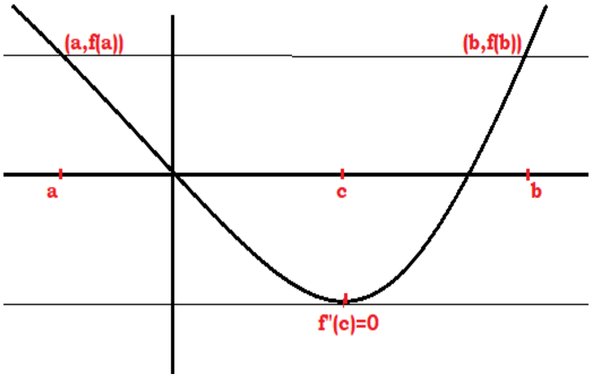
$$f(2\pi) = 2\pi - 2\sin\pi = 2\pi = 6.2$$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - 2\frac{\sqrt{3}}{2} = -0.68$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + 2\frac{\sqrt{3}}{2} = 6.96$$

From the above, the absolute maximum exists at $x = \frac{5\pi}{3}$ and its maximum value is 6.96 and the absolute minimum exists at $x = \frac{\pi}{3}$ and its minimum value is -0.68.

Rolle's Theorem



If a function f(x) is continuous in [a,b], differentiable in (a,b) and f(a) = f(b) then there exists at least one real number c$\in$(a,b) such that f'(c)=0.

1. **Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root in [0,1]**

First we use Intermediate Value Theorem, to prove a root exists in the interval.

Let $f(x) = x^3 + x - 1$.

Here $f(0) = 0 + 0 - 1 = -1 < 0 \quad and \quad f(1) = 1 + 1 - 1 = 1 > 0$

Since $f(x)$ is polynomial, it is continuous. Hence by Intermediate Value theorem, there exists a number c between 0 and 1 such that $f(c) = 0$. Therefore the given equation has a root.

To show that the equation has no other real root in the interval, we use Rolle's theorem.

Suppose there exists two roots $a \; and \; b$. Then $f(a) = 0 = f(b)$.

Since $f(x)$ is polynomial, it is differentiable in (a,b) and continuous in [a,b].

Then by Rolle's theorem, there exists a number c between $a \; and \; b$ such that $f'(c) = 0$.

But $f'(x) = 3x^2 + 1 \geq 0 \; for \; all \; x$. i.e. $f'(x)$ can never be 0. This is a contradiction to the theorem.

Therefore the equation cannot have two roots.

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2. **Let $f(x)=1-x^{\frac{2}{3}}$. Show that $f(-1)=f(1)$ but there is no number c in $(-1,1)$ such that $f'(c)=0$. Why does this is not contradicts Rolle's theorem.**

Given $f(x)=1-x^{\frac{2}{3}}$

$f(1)=1-1^{\frac{2}{3}}=0$ *and* $f(-1)=1-(-1)^{\frac{2}{3}}=1-1=0$

Now $f'(x)=-\frac{2}{3}x^{-\frac{1}{3}}=-\frac{2}{3}\frac{1}{x^{\frac{1}{3}}}$. But $f'(c)\neq 0$ for any c in $(-1,1)$.

But this is not contradiction to the Rolle's theorem, because $f'(0)$ does not exist and hence f is not differentiable in $(-1,1)$.

3. **Verify Rolle's theorem for f(x)=3x⁴–4x²+5 in [–1,1].**

Given f(x) = $3x^4 - 4x^2 + 5$.

Clearly f(x) continuous in (–1,1)and derivable in [–1,1]

Also f(–1) = f(1) = 4. Therefore conditions of Rolle's theorem holds good.

f'(c) = 0 gives $12c^3 - 8c = 0$.

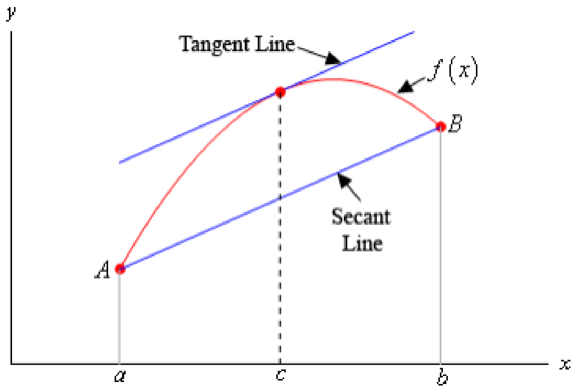
$4c\,(3c^2 - 2) = 0$

i.e. c = 0, $c^2 = 2/3$, c =±√(2/3)

Here –1 < 0 <1, –1 <√(2/3) <1, –1 <–√(2/3) < 1

The Mean Value Theorem

If a function f(x) is continuous in [a,b], differentiable in (a,b) and f(a) ≠ f(b) then there exists at least one real number c∈(a,b) such that $\frac{f(b)-f(a)}{b-a}=f'(c)$.



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1. **Find c of mean value theorem for $x^3 + x$ in $[1,2]$.**

Given f(x) = x^3+ x. Clearly f(x) continuous in [1,2] and derivable in (1,2).

f'(x) = $3x^2$ + 1. Therefore conditions of mean value theorem holds good.

Also f(1) = 2 and f(2) = 10

$f'(c) = \frac{f(b) - f(a)}{b - a}$ i.e. $f'(c) = \frac{f(2) - f(1)}{2-1} = \frac{10-2}{1} = 8$ i.e. $3c^2 + 1 = 8$

$f'(c) = 0$ gives $3c^2 = 7$

Solving, we get $c = \pm\sqrt{\frac{7}{3}}$. But $c = \pm\sqrt{\frac{7}{3}}$ ∈ [1,2].

2. **Let $f(x)$ is continuous and differentiable in [0,2]. Also $f(0) = -3$ *and* $f'(x) \le 5, \forall x.$ What is the largest possible value for $f(2)$?**

Given that f is differentiable everywhere. We apply Mean Value theorem in the interval $[0,2]$.

There exists a number c such that $\frac{f(2) - f(0)}{2-0} = f'(c)$

$$f(2) - f(0) = 2f'(c)$$

$$f(2) + 3 = 2f'(c)$$

$$f(2) = -3 + 2f'(c)$$

$$f(2) \le -3 + 10, \quad But\ f'(c) \le 5,\ i.e.\ 2f'(c) \le 10$$

$$f(2) \le 7$$

∴ the largest possible value of $f(2)$ is 7.

3 **Let $f(x)$ is continuous and differentiable in [6,15]. Also $f(6) = -2$ *and* $f'(x) \le 10$ for all x. What is the largest possible value for $f(15)$?**

Given that f is differentiable everywhere. We apply Mean Value theorem in the interval $[6,15]$.

There exists a number c such that $\frac{f(15) - f(6)}{15-6} = f'(c)$

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$$f(15)-f(6)=9f'(c)$$

$$f(15)+2=9f'(c)$$

$$f(15)=-2+9f'(c)$$

$$f(15)\le -2+90, \quad But\ f'(c)\le 10,\ i.e.\ 9f'(c)\le 90$$

$$f(15)\le 88$$

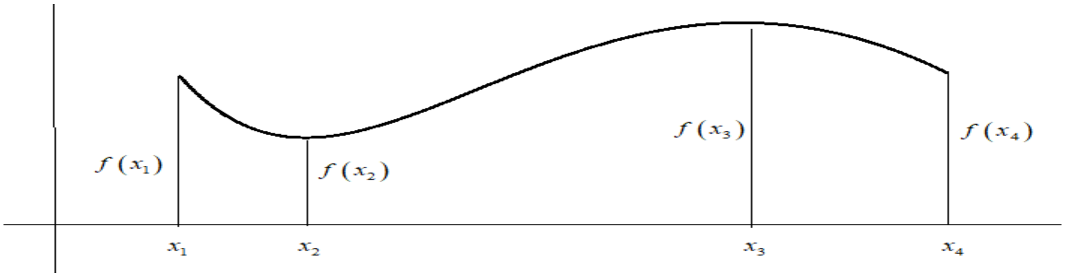
$\therefore$ the largest possible value of $f(15)$ is 88.

Increasing Decreasing Functions

A function $f(x)$ is called increasing in an interval I if $f(x_1)<f(x_2)$ whenever $x_1<x_2$ in I .
A function $f(x)$ is called decreasing in an interval I if $f(x_1)>f(x_2)$ whenever $x_1<x_2$ in I .

Note:

1 Derivative of an increasing function is positive and the derivative of a decreasing function is negative.
2 Graphically, a function is increasing on an interval if its slopes upward to the right and decreasing on an interval if its slopes downward to the right.
3 If the derivative is 0, then the function is neither increasing nor decreasing.



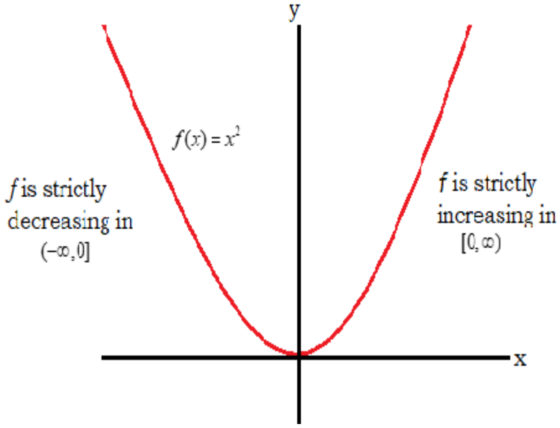
Here the function is decreasing in the interval (x_1, x_2) and increasing in the interval (x_2, x_3) and again decreasing in the interval (x_3, x_4).

Increasing / Decreasing Test

Let f be continuous in an interval I and differentiable at interior points of I. Then

(i) If $f'(x)>0$ on interior points of an interval, then f is increasing on that interval. f is strictly increasing if $f'(x)=0$ for at most finite number of points in I.

(ii) If $f'(x)<0$ on interior points of an interval, then f is decreasing on that interval. f is strictly decreasing if $f'(x)=0$ for at most finite number of points in I.



Example: On which interval is $f(x)=2x^3-6x^2+6x-7$ strictly increasing or decreasing?

Let $f'(x)=6x^2-12x+6=6(x-1)^2$

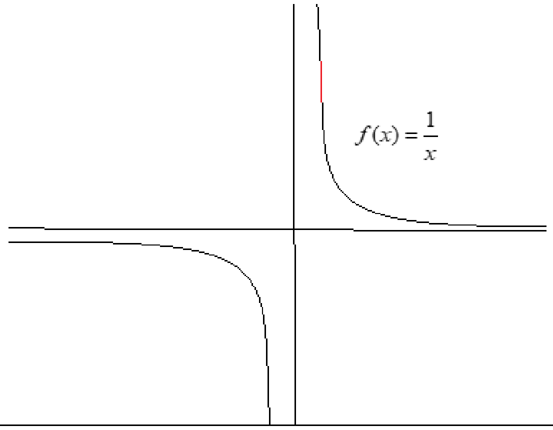
Here $f'(x)>0 \quad \textit{for all } x \textit{ except at } x=1$

When $x=1, \quad f'(x)=0$.

Hence by definition, $f(x)$ is strictly increasing in $(-\infty,\infty)$.

Example: Consider $f(x)=\dfrac{1}{x}$. Even though $f'(x)=-\dfrac{1}{x^2}<0$

it is not strictly decreasing function in its domain. From the figure, It is strictly decreasing in each of the intervals $(-\infty,0)$ *and* $(0,\infty)$.



THE FIRST DERIVATIVE TEST

Suppose that c is a critical number of a continuous function f defined in an interval I.

(a) If f' changes from positive to negative at c, then f has a local(relative) maximum at c.

(b) If f' changes from negative to positive at c, then f has a local(relative) minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

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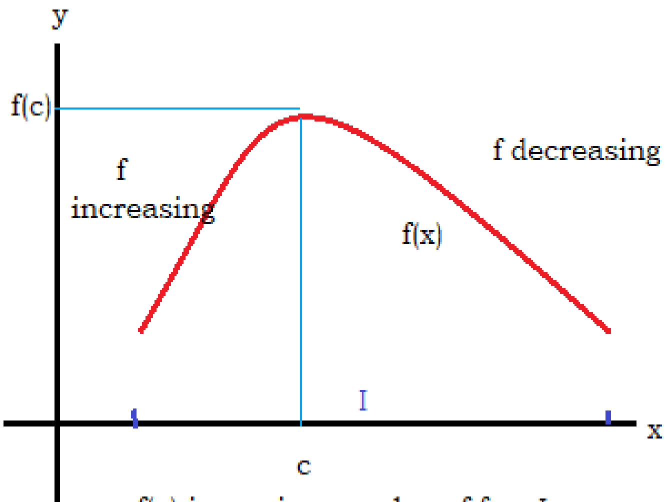
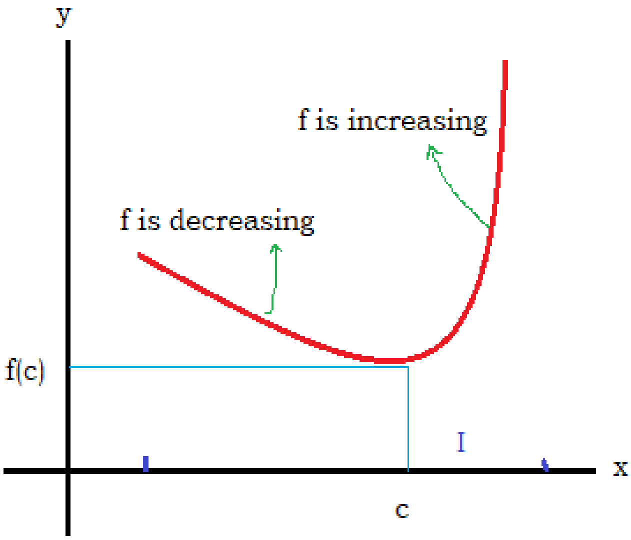


f(c) is maximum value of f on I



f(c) is minimum value of f on I

Solved Problems

1. Apply first derivative test to find the local maximum and local minimum of values of the function $f(x)=(x-1)^2(x-3)^2$.

Given $f(x)$ is continuous in the interval $(-\infty,\infty)$

$$f'(x)=2(x-1)(x-3)^2+2(x-1)^2(x-3)$$

$$=2(x-1)(x-3)\left[(x-3)+(x-1)\right]$$

$$=2(x-1)(x-2)(x-3)$$

Since $f'(x)$ exists for all x, the only critical values are given by $f'(x)=0$.

$(x-1)(x-2)(x-3)=0$

Therefore $x=1,\ 2,\ 3$ are the critical points.

Interval	$f'(x)=2(x-1)(x-2)(x-3)$	$f(x)$
$-\infty<x<1$	–	Decreasing on $(-\infty,1)$
$1<x<2$	+	Increasing on $(1,2)$
$2<x<3$	–	Decreasing on $(2,3)$
$3<x<\infty$	+	Increasing on $(3,\infty)$

Since $f'(x)$ changes from negative to positive at 1, the first derivative test gives that there is a local minimum at 1 and the local minimum value is $f\left(1\right)=0$.

Since $f'(x)$ changes from positive to negative at 2, the first derivative test gives that there is a local maximum at 2 and the local maximum value is $f\left(2\right)=1$.

Since $f'(x)$ changes from negative to positive at 3, the first derivative test gives that there is a local minimum at 3 and the local minimum value is $f\left(3\right)=0$.

2. Find the local maximum and local minimum of values of the function $f(x)=x+2\sin x,\ 0\le x\le 2\pi.$

Given $f(x)$ is continuous in the closed interval $\left[0,2\pi\right]$.

$f'(x)=1+2\cos x$. Since $f'(x)$ exists for all x, the only critical values are given by $f'(x)=0$.

$$f'(x)=0 \Rightarrow 1+2\cos x=0$$

$$i.e.\quad \cos x=-\frac{1}{2}$$

$$\therefore x=\frac{2\pi}{3},\frac{4\pi}{3}\ are\ the\ critical\ po\text{int}s.$$

Interval	$f'(x)=1+2\cos x$	$f(x)$
$0<x<\frac{2\pi}{3}$	+	Increasing on $\left(0,\frac{2\pi}{3}\right)$
$\frac{2\pi}{3}<x<\frac{4\pi}{3}$	–	Decreasing on $\left(\frac{2\pi}{3},\frac{4\pi}{3}\right)$
$\frac{4\pi}{3}<x<2\pi$	+	Increasing on $\left(\frac{4\pi}{3},2\pi\right)$

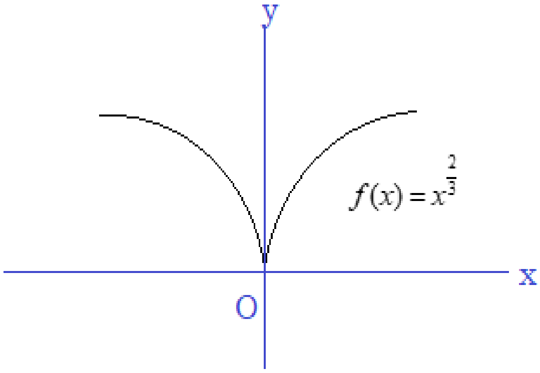
Since $f'(x)$ changes from positive to negative at $\frac{2\pi}{3}$, the first derivative test gives that there is a local maximum at $\frac{2\pi}{3}$ and the local maximum value is $f\left(\frac{2\pi}{3}\right)=\frac{2\pi}{3}+2\sin\frac{2\pi}{3}=\frac{2\pi}{3}+2\frac{\sqrt{3}}{2}$.

Since $f'(x)$ changes from negative to positive at $\frac{4\pi}{3}$, the first derivative test gives that there is a local minimum at $\frac{4\pi}{3}$ and the local minimum value is $f\left(\frac{4\pi}{3}\right)=\frac{4\pi}{3}+2\sin\frac{4\pi}{3}=\frac{4\pi}{3}-2\frac{\sqrt{3}}{2}$.

Special Case

https://doi.org/10.5281/zenodo.15288097

It can be conclude from the first derivative test that extreme value exists at a critical point c even if $f'(c)$ does not exist.



Let $f(x)=x^{\frac{2}{3}}$, which is continuous in $(-\infty,\infty)$.

Then $f'(x)=\frac{2}{3}x^{-\frac{1}{3}}=\frac{2}{3}\frac{1}{x^{\frac{1}{3}}}$ for $x\neq 0$

Here $f'(0)$ does not exist.

Also $f'(x)<0 \quad for \quad x<0$ and $f'(x)>0 \quad for \quad x>0$

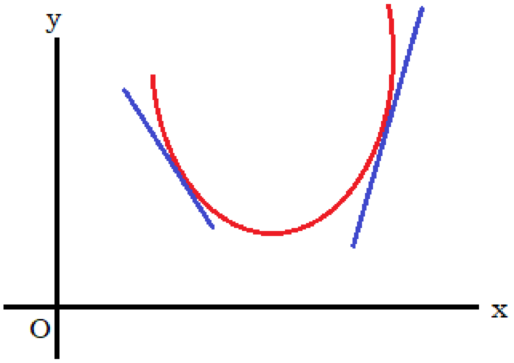
Thus $f'(x)$ changes from negative to positive at 0.

$\therefore$ By first derivative test, $f(0)=0$ is the relative minimum of f even though $f'(0)$ does not exist.

CONCAVITY AND INFLECTION POINTS

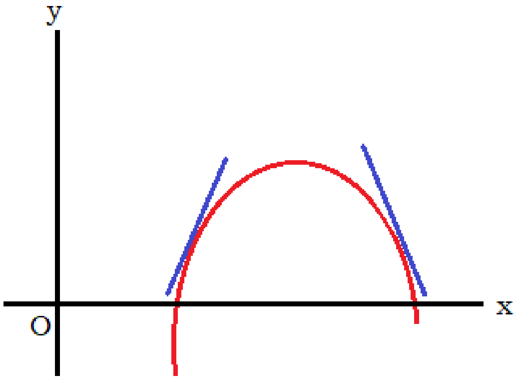
If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I.

If $f''(x)>0$ for all x in I, then the graph of f is concave upward on I.



If the graph of f lies below all of its tangents on I, it is called concave downward on I.

If $f''(x)<0$ for all x in I, then the graph of f is concave downward on I.



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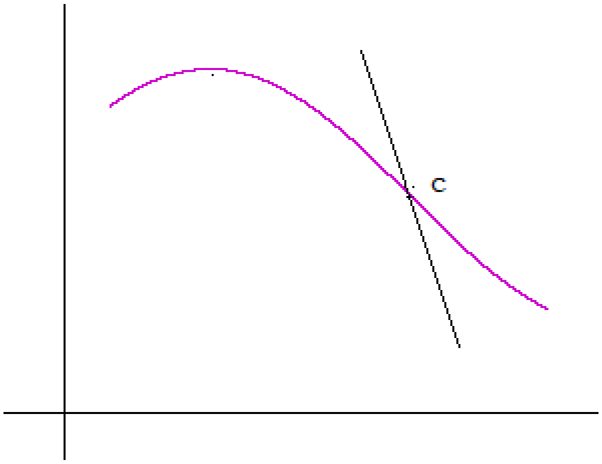
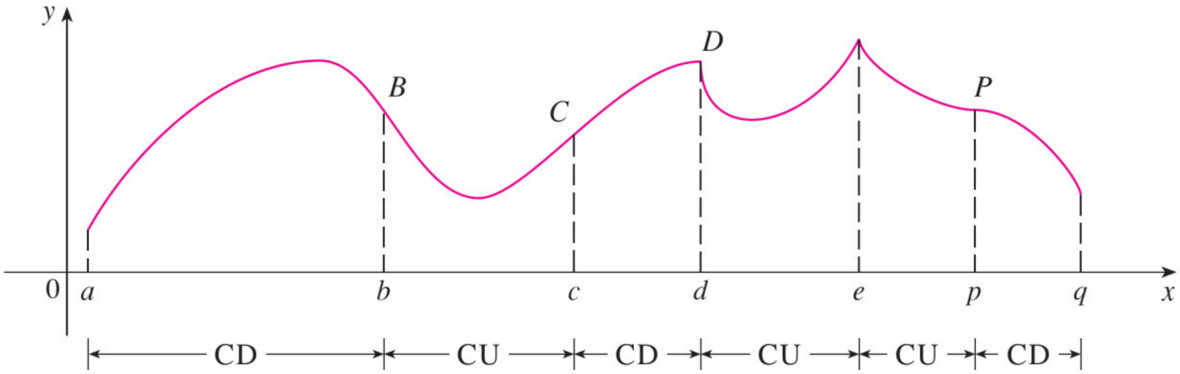
If two parts of the curve lie on different sides of the tangent at a point, the point is said to be point of inflection where $f''(x)=0 \quad and \quad f'''(x)\neq 0.$



The following figure shows the graph of a function that is concave upward (abbreviated CU) on the intervals (b,c) , (d,e) , and (e,p) and concave downward (CD) on the intervals (a,b) , (c,d) , and (p,q) .



DEFINITION

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

In the above figure the points B, C, D and P are points of inflection.

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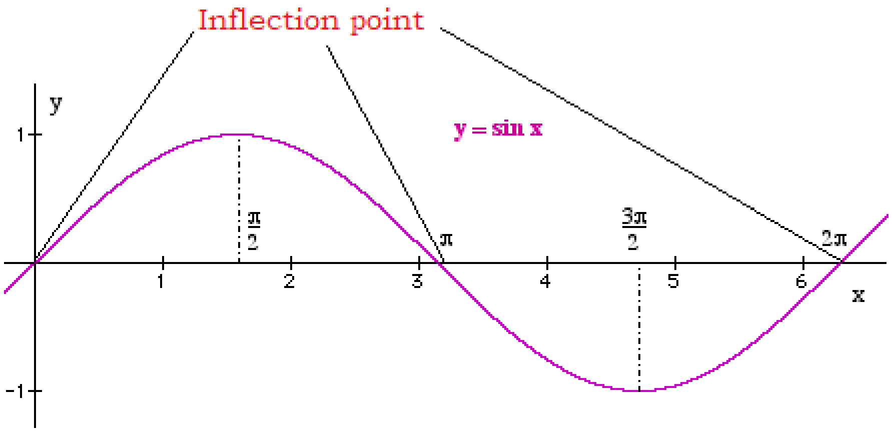


Solved Problems

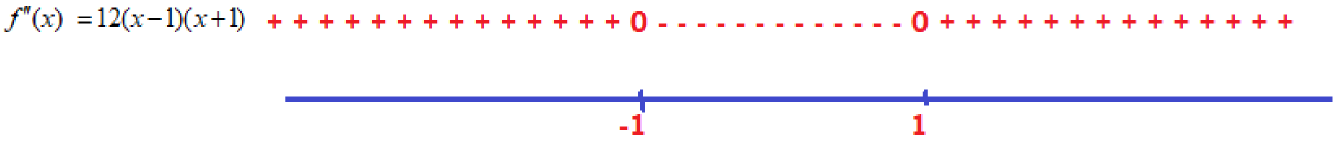
1 Let $f(x) = x^2 - 6x^2 + 8x + 10$. Find the points of inflection of f.

The first and second derivatives are

$$f'(x) = 4x^3 - 12x + 8 \quad \text{and} \quad f''(x) = 12x^2 - 12 = 12(x-1)(x+1)$$

$$f''(x) = 0 \Rightarrow (x-1)(x+1) = 0 \quad i.e. \quad x = -1,\ 1$$

Let us find the sign of $f''(x)$ in the interval $(-\infty, -1)$, $(-1, 1)$ *and* $(1, \infty)$



Here $f''(1) = f''(-1) = 0$. Also $f''(x)$ changes sign at both 1 and -1.

Therefore $f(1) = 13$, *and* $f(-1) = -3$.

Hence $(1, 13)$ *and* $(-1, -3)$ are the inflection points of f .

2 Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

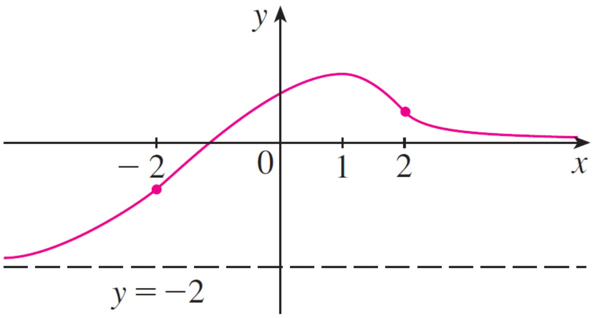
(ii) $f''(x) > 0 on (-\infty, -2) and (2, \infty)$, $f''(x) < 0 on (-2, 2)$

(iii) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

Condition(i) tells us that f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

Condition (ii) says that f is concave upward on $(-\infty,-2)$ and $(2,\infty)$, and concave downward on $(-2,2)$.

From condition (iii) we know that the graph of f has two horizontal asymptotes: $y=-2$ and $y=0$.



We first draw the horizontal asymptote $y=-2$ as a dashed line. We then draw the graph of f approaching this asymptote at the far left, increasing to its maximum point at $x=1$ and decreasing toward the x-axis at the far right.

We also make sure that the graph has inflection points when $x=-2$ and 2. Notice that we made the curve bend upward for $x<-2$ and $x>2$, and bend downward when x is between -2 and 2.

3 Sketch the graph of $f(x)=\dfrac{2x+1}{3x-1}$**.**

First we find the derivatives of f :

$$f'(x)=\frac{(3x-1)(2)-(2x+1)(3)}{(3x-1)^2}=\frac{-5}{(3x-1)^2}$$

$$f''(x)=\frac{30}{(3x-1)^3}$$

Here $f'(x)\neq 0$ for all x. But $f'(x)$ does not exist for $x=\frac{1}{3}$. $\therefore$ the only critical point is $x=\frac{1}{3}$

Interval	sign of $f'(x)$	$f(x)$	sign of $f''(x)$	Concavity
$-\infty<x<\frac{1}{3}$	Negative	Decreasing	Negative	Concave downward
$\frac{1}{3}<x<\infty$	Negative	Decreasing	Positive	concave upward

Also $\underset{x\to\frac{1^+}{3}}{Lt} f(x)=\underset{x\to\frac{1^+}{3}}{Lt}\frac{2x+1}{3x-1}=\infty$ and $\underset{x\to\frac{1^-}{3}}{Lt} f(x)=\underset{x\to\frac{1^-}{3}}{Lt}\frac{2x+1}{3x-1}=-\infty$

Therefore $x=\frac{1}{3}$ is the vertical asymptote.

Also $\underset{x\to\infty}{Lt} f(x)=\underset{x\to\infty}{Lt}\frac{2x+1}{3x-1}=\underset{x\to\infty}{Lt}\frac{x\left(2+\frac{1}{x}\right)}{x\left(3-\frac{1}{x}\right)}=\frac{2}{3}$. Similarly $\underset{x\to-\infty}{Lt} f(x)=\frac{2}{3}$

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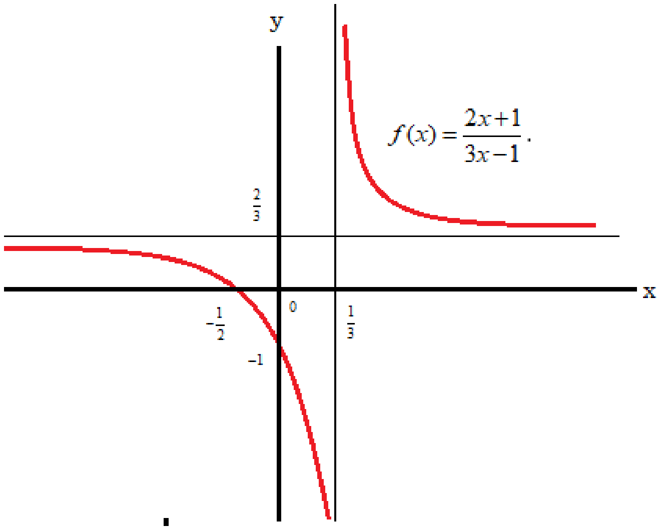
Therefore $y = \frac{2}{3}$ is a horizontal asymptote. From these information, we can draw the graph of f.



The Second Derivative Test

Suppose f'' is continuous near c.

(i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(ii) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Note: If $f''(c) = 0$, we cannot derive any conclusions about relative extremum of f at c.

1. Find the intervals of concavity and the points of inflection for the curve $y = 2x^3 + 3x^2 - 36x$.

Given $f(x) = 2x^3 + 3x^2 - 36x$. Then $f'(x) = 6x^2 + 6x - 36$ *and* $f''(x) = 12x + 6$

The critical points are given by $f'(x) = 0$

$$6x^2 + 6x - 36 = 0$$
$$x^2 + x - 6 = 0$$
$$(x-2)(x+3) = 0$$
$$\therefore\ x = 2,\ x = -3$$

$f''(x) = 0$ gives $12x + 6 = 0$, *i.e.* $x = -\frac{1}{2}$. Hence we divide the real line into two intervals with this numbers as end points.

Interval	$f''(x) = 12x + 6$	Concavity
$\left(-\infty, -\frac{1}{2}\right)$	–	Downward
$\left(-\frac{1}{2}, \infty\right)$	+	Upward

The point $x=-\frac{1}{2}$ is an inflection point since the curve changes from concave downward to concave upward.

2 **Discuss the curve $y=x^4-4x^3$ with respect to concavity, points of inflection, local maxima and minima.**

Given $f(x)=x^4-4x^3$. Then $f'(x)=4x^3-12x^2$ *and* $f''(x)=12x^2-24x$

The critical points are given by $f'(x)=0$

$$4x^3-12x^2=0$$
$$4x^2(x-3)=0$$
$$\therefore x=0,\ x=3$$

We calculate f'' at these critical points

$$f''(x)=12x^2-24x$$
$$f''(0)=12(0)-24(0)=0$$
$$f''(3)=12(9)-24(3)=36>0$$

Since $f'(3)=0$ *and* $f''(3)>0$, by second derivative test, f has local minimum at $x=3$ and the minimum value is $f(3)=3^4-4\times3^3=-27$

Second derivative test gives no information about the critical point $x=0$, since $f''(0)=0$

But since $f'(x)<0$ *for* $x<0$ *and* $0<x<3$, first derivative test tells us that f does not have a local maximum or minimum at $x=0$.

$f''(x)=0$ gives $12x^2-24x=0,$ *i.e.* $12x(x-2)=0$ gives $x=0,\ 2$. Hence we divide the real line into intervals with these numbers as end points.

Interval	$f''(x)=12x^2-24x$	Concavity
$(-\infty,0)$	+	Upward
$(0,2)$	–	Downward
$(2,\infty)$	+	Upward

The point $x=0$ is an inflection point since the curve changes from concave upward to concave downward.

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The point $x=2$ is an inflection point since the curve changes from concave downward to concave upward.

3 Find where the function $f(x)=3x^4-4x^3-12x^2+5$ is increasing or decreasing. Find the local maximum and local minimum values of f.

(i) Given $f(x)=3x^4-4x^3-12x^2+5$ **and**

$f'(x)=12x^3-12x^2-24x=12x\left(x^2-x-2\right)=12x(x-2)(x+1)$

Interval	(x)	$(x-2)$	$(x+1)$	$f'(x)$	$f(x)$
$x<-1$	–	–	–	–	Decreasing on $(-\infty,-1)$
$-1<x<0$	–	–	+	+	Increasing on $(-1,0)$
$0<x<2$	+	–	+	–	Decreasing on $(0,2)$
$x>2$	+	+	+	+	Increasing on $(2,\infty)$

(ii) f changes from decreasing to increasing at $x=-1$ and hence f has local minimum at $x=-1$ and the minimum value is $f(-1)=3(-1)^4-4(-1)^3-12(-1)^2+5=3+4-12+5=0$

f changes from increasing to decreasing at $x=0$ and hence f has local maximum at $x=0$ and the maximum value is $f(0)=3(0)^4-4(0)^3-12(0)^2+5=5$

f changes from decreasing to increasing at $x=2$ and hence f has local minimum at $x=2$ and the minimum value is $f(2)=3(2)^4-4(2)^3-12(2)^2+5=48-32-48+5=-27$

4. For the function $f(x)=2x^3+3x^2-36x$, (i) Find the intervals on which it is increasing or decreasing (ii) Find the local maximum and minimum values of f. (iii) Find the intervals of concavity and the inflection points

(i) Given $f(x)=2x^3+3x^2-36x$ **and** $f'(x)=6x^2+6x-36=6(x+3)(x-2)$

Interval	$(x+3)$	$(x-2)$	$f'(x)$	$f(x)$
$x<-3$	–	–	+	Increasing on $(-\infty,-3)$
$-3<x<2$	+	–	–	Decreasing on $(-3,2)$
$x>2$	+	+	+	Increasing on $(2,\infty)$

(ii) f changes from increasing to decreasing at $x=-3$ and hence f has local maximum at $x=-3$ and the maximum value is $f(-3)=-2\times 27+3\times 9+36\times 3=81$

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f changes from decreasing to increasing at $x=2$ and hence f has local minimum at $x=2$ and the minimum value is $f(2)=2\times 8+3\times 4-36\times 2=-44$

(iii) $f''(x)=12x+6=6(2x+1)$ $f''(x)=0$ gives $(2x+1)=0$, *i.e.* $x=-\frac{1}{2}$. Hence we divide the real line into intervals with these numbers as end points.

Interval	$f''(x)=2x+1$	Concavity
$\left(-\infty,-\frac{1}{2}\right)$	–	Downward
$\left(-\frac{1}{2},\infty\right)$	+	Upward

The point $x=-\frac{1}{2}$ is an inflection point since the curve changes from concave downward to concave upward.

5. Find the local maximum and minimum values of $f(x)=\sqrt{x}-\sqrt[4]{x}$ using both the first and second derivative tests.

First Derivative Test	Second Derivative Test
$f(x)=\sqrt{x}-\sqrt[4]{x}=x^{\frac{1}{2}}-x^{\frac{1}{4}}$ $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{4}x^{-\frac{3}{4}}=\frac{1}{2}\frac{1}{x^{\frac{1}{2}}}-\frac{1}{4}\frac{1}{x^{\frac{3}{4}}}$ $=\frac{2x^{\frac{1}{4}}-1}{4x^{\frac{3}{4}}}$ *Let* $f'(x)=0$, *then* $\frac{2x^{\frac{1}{4}}-1}{4x^{\frac{3}{4}}}=0$ $2x^{\frac{1}{4}}-1=0$ $x^{\frac{1}{4}}=\frac{1}{2}$ $x=\frac{1}{16}$ *is the critical point* Also $f'(x)$ does not exist at $x=0$, which is also a critical point.	$f(x)=\sqrt{x}-\sqrt[4]{x}=x^{\frac{1}{2}}-x^{\frac{1}{4}}$ $f'(x)=\frac{1}{2}\frac{1}{x^{\frac{1}{2}}}-\frac{1}{4}\frac{1}{x^{\frac{3}{4}}}=\frac{2x^{\frac{1}{4}}-1}{4x^{\frac{3}{4}}}$ *Let* $f'(x)=0$, *then* $\frac{2x^{\frac{1}{4}}-1}{4x^{\frac{3}{4}}}=0$ $2x^{\frac{1}{4}}-1=0$ $x^{\frac{1}{4}}=\frac{1}{2}$ $x=\frac{1}{16}$ *is the critical point* $f''(x)=-\frac{1}{2}\frac{1}{2}x^{-\frac{3}{2}}+\frac{3}{4}\frac{1}{4}x^{-\frac{7}{4}}$ $=-\frac{1}{2}\frac{1}{2}\frac{1}{x^{\frac{3}{2}}}+\frac{3}{4}\frac{1}{4}\frac{1}{x^{\frac{7}{4}}}$

Interval	$f'(x)=\frac{1}{2}\frac{1}{x^{\frac{1}{2}}}-\frac{1}{4}\frac{1}{x^{\frac{3}{4}}}$	$f(x)$
$-\infty<x<0$	does not exist	
$0<x<\frac{1}{16}$	$-$	Decreasing
$x>\frac{1}{16}$	$+$	Increasing

Since $f'(x)$ changes from negative to positive at $\frac{1}{16}$, the first derivative test gives that there is a local minimum at $\frac{1}{16}$ and the local minimum value is

$$f\left(\frac{1}{16}\right)=\left(\frac{1}{16}\right)^{\frac{1}{2}}-\left(\frac{1}{16}\right)^{\frac{1}{4}}$$
$$=\frac{1}{4}-\frac{1}{2}$$
$$=-\frac{1}{4}$$

$$f''\left(\frac{1}{16}\right)=-\frac{1}{2}\frac{1}{2}\frac{1}{\left(\frac{1}{16}\right)^{\frac{3}{2}}}+\frac{3}{4}\frac{1}{4}\frac{1}{\left(\frac{1}{16}\right)^{\frac{7}{4}}}$$
$$=-\frac{1}{4}64+\frac{3}{16}128=8>0$$

Since $f'\left(\frac{1}{16}\right)=0$ *and* $f''\left(\frac{1}{16}\right)>0$, by second derivative test, f has local minimum at $x=\frac{1}{16}$ and the minimum value is

$$f\left(\frac{1}{16}\right)=\left(\frac{1}{16}\right)^{\frac{1}{2}}-\left(\frac{1}{16}\right)^{\frac{1}{4}}$$
$$=\frac{1}{4}-\frac{1}{2}$$
$$=-\frac{1}{4}$$

6 For the function $f(x)=x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$

(i) Find the intervals on which it is increasing or decreasing

(ii) Find the local maxima and minima of f

(iii) Find the intervals of concavity and the inflection points

Given $f(x)=x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$

Therefore

$$f'(x)=\frac{2}{3}x^{-\frac{1}{3}}(6-x)^{\frac{1}{3}}-\frac{1}{3}x^{\frac{2}{3}}(6-x)^{-\frac{2}{3}}$$
$$=x^{-\frac{1}{3}}(6-x)^{-\frac{2}{3}}\left[\frac{2}{3}(6-x)-\frac{1}{3}x\right]$$
$$=x^{-\frac{1}{3}}(6-x)^{-\frac{2}{3}}\left[4-x\right]$$
$$=\frac{4-x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$$

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Here $f'(x) = 0$ when $x = 4$.

Also $f'(x)$ does not exist at $x = 0$ & 6. Therefore the critical points are 0, 4, 6.

Interval	$4-x$	$x^{\frac{1}{3}}$	$(6-x)^{\frac{2}{3}}$	$f'(x)$	$f(x)$
$-\infty < x < 0$	+	–	+	–	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	–	+	+	–	decreasing on $(4, 6)$
$6 < x < \infty$	–	+	+	–	decreasing on $(6, \infty)$

Since $f'(x)$ changes from negative to positive at 0, the first derivative test gives that there is a local minimum at 0 and the local minimum value is $f(0) = 0^{\frac{2}{3}}(6-0)^{\frac{1}{3}} = 0$

Since $f'(x)$ changes from positive to negative at 4, the first derivative test gives that there is a local maximum at 4 and the local maximum value is $f(4) = 4^{\frac{2}{3}}(6-4)^{\frac{1}{3}} = 2^{\frac{5}{3}}$

Since $f'(x)$ does not change sign at $x = 6$, there is no maximum or minimum at this point.

$$\begin{aligned}
f''(x) &= -x^{-\frac{1}{3}}(6-x)^{-\frac{2}{3}} - \frac{1}{3}x^{-\frac{4}{3}}(6-x)^{-\frac{2}{3}}[4-x] + \frac{2}{3}(6-x)^{-\frac{5}{3}}x^{-\frac{1}{3}}[4-x] \\
&= x^{-\frac{4}{3}}(6-x)^{-\frac{5}{3}}\left[-x(6-x) - \frac{1}{3}(6-x)(4-x) + \frac{2}{3}x(4-x)\right] \\
&= x^{-\frac{4}{3}}(6-x)^{-\frac{5}{3}}\left[-6x + x^2 - 8 + \frac{10x}{3} - \frac{x^2}{3} + \frac{8x}{3} - \frac{2x^2}{3}\right] \\
&= x^{-\frac{4}{3}}(6-x)^{-\frac{5}{3}}[-8] \\
&= -\frac{8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}
\end{aligned}$$

$f''(x) = -\frac{8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$. From the expression we see that $f''(x)$ changes sign near 0 *and* $6.$. Hence we divide the real line into intervals with these numbers as end points.

Interval	$f''(x)$	Concavity
$(-\infty, 0)$	–	Downward

$(0,6)$	$-$	Downward
$(6,\infty)$	+	Upward

The point $x=6$ is an inflection point since the curve changes from concave downward to concave upward.

7 **Find the maximum and minimum values of** $2x^3-3x^2-36x+10$

Let $f(x)=2x^3-3x^2-36x+10$

$f'(x)=6x^2-6x-36$ and $f''(x)=12x-6$

$$f'(x)=0 \text{ gives } 6x^2-6x-36=0$$
$$i.e.\ x^2-x-6=0$$
$$(x+2)(x-3)=0$$
$$x=-2,\ 3$$

At $x=-2$, $f''(x)=12(-2)-6=-30$, *negative.*

Hence f has maximum and the maximum value is

$$f(-2)=2(-8)-3(4)-36(-2)+10$$
$$=-16-12+72+10=54$$

At $x=3$, $f''(x)=12(3)-6=30$, *positive.*

Hence f has minimum and the minimum value is

$$f(3)=2(27)-3(9)-36(3)+10$$
$$=54-27-108+10=-71$$

8. **Estimate the maxima and minima of the function** $10x^6-24x^5+15x^4-40x^3+108$

Let $f(x)=10x^6-24x^5+15x^4-40x^3+108.$

$f'(x)=60x^5-120x^4+60x^3-120x^2$ and $f''(x)=300x^4-480x^3+180x^2-240x$

$$f'(x)=0 \quad gives \quad 60x^5-120x^4+60x^3-120x^2=0$$

$$60x^2\left(x^3-2x^2+x-2\right)=0$$

$$x=0,0 \ \ and \ \left(x^3-2x^2+x-2\right)=0 \quad hence \quad x=2$$

At $x=0$, $f''(0)=0$ and hence x=0 is a point of inflection

At $x=2$, $f''(2)=4800-3840+720-480=positive$. Therefore f(x) has minimum at x=2.

The minimum value is $f(2)=640-768+240-320+108=-100.$

9. Estimate the local extrema of $x^4 - 8x^2$ using second derivative test.

Let $f(x) = x^4 - 8x^2$

$$f'(x) = 4x^3 - 16x \quad \text{and} \quad f''(x) = 12x^2 - 16$$

$$f'(x) = 0 \quad gives \quad 4x(x^2 - 16) = 0$$

$$x = 0, 4, -4$$

At $x = 0, \quad f''(0) = -16, \ negative$ and hence f(x) has maximum and the maximum value is f(0) = 0.

At $x = 4, \quad f''(4) = 196 - 16 = positive$. Therefore f(x) has minimum at x=4 and the

minimum value is $f(4) = 4^4 - 8(4)^2 = 128.$

At $x = -4, \quad f''(-4) = 192 - 16 = positive$. Therefore f(x) has minimum at x=–4 and the

minimum value is $f(-4) = (-4)^4 - 8(-4)^2 = 128.$

10. Divide 20 into two parts so that the product of the square of the one and cube of the other may be the greatest possible.

Let the two parts be $x \,\&\, y$ so that $x + y = 20 \,........(i)$

Let $z = y^2 x^3$

$$z = (20 - x)^2 x^3 \quad \{\text{using (i)}\}$$

$$z = (400 - 40x + x^2)x^3 = 400x^3 - 40x^4 + x^5$$

$$\frac{dz}{dx} = 1200x^2 - 160x^3 + 5x^4$$

$$\frac{dz}{dx} = 0 \Rightarrow 1200x^2 - 160x^3 + 5x^4 = 0$$

$$5x^2\left(240 - 32x + x^2\right) = 0$$

$$5x^2\left(x - 20\right)\left(x - 12\right) = 0$$

$\therefore \ x = 0, 12, 20$ are the stationary points. But x cannot be 0 or 20.

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Let $x = 12$

$$\frac{d^2z}{dx^2} = 2400x - 480x^2 + 20x^3$$

At $x = 12,\quad \frac{d^2z}{dx^2} = 2400(12) - 480(12)^2 + 20(12)^3 < 0$

$\therefore\ at\ x = 12,\ z = y^2x^3$ is maximum.

Therefore the two parts into which 20 can be divided are 12 and 8.

11. Find the maximum value and minimum value of the function $y = (x-2)^2(x-3).$

Given $y = (x-2)^2(x-3).$ $\frac{dy}{dx} = (x-2)(3x-8).$ and $\frac{d^2y}{dx^2} = 6x - 14$

$\frac{dy}{dx} = 0\ gives\ x = 2\quad and\quad x = \frac{8}{3}$

$When\ x = 2,\quad \frac{d^2y}{dx^2} = 12 - 14 = -2,\ negative$

Therefore, when x = 2, y has maximum and the maximum value is y = 0.

$When\ x = \frac{8}{3},\quad \frac{d^2y}{dx^2} = 6 \times \frac{8}{3} - 14 = 2,\ positive$

Therefore, when x = $\frac{8}{3}$, y has minimum and the minimum value is

$$y = \left(\frac{8}{3} - 2\right)^2\left(\frac{8}{3} - 3\right) = -\frac{4}{27}$$

12. Find the relative(local) extrema for the function $f(x) = x^3 + x^2 - 8x - 1$

Given $f(x) = x^3 + x^2 - 8x - 1$ and $f'(x) = 3x^2 + 2x - 8 = (3x-4)(x+2)$

$f'(x) = 0\ gives\ x = -2\ and\ x = \frac{4}{3}$

Critical Point		Interval	Value of $f'(x)$
Part I	$x = \frac{4}{3}$	$-2 < x < \frac{4}{3}$	$f'(x) < 0$
	$x = \frac{4}{3}$	$x > \frac{4}{3}$	$f'(x) > 0$
Part II	$x = -2$	$x < -2$	$f'(x) > 0$
	$x = -2$	$-2 < x < \frac{4}{3}$	$f'(x) < 0$

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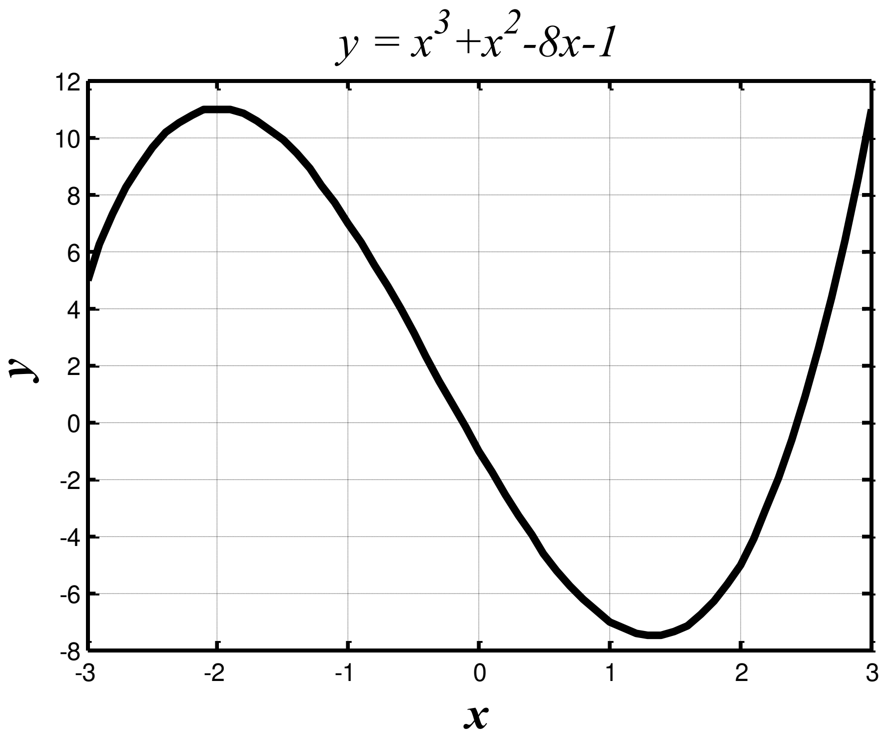
In part I, sign changes between negative to positive, by first derivative, a local minimum exist at $x=\frac{4}{3}$ and the minimum value is $f\left(\frac{4}{3}\right)=-\frac{203}{27}$

In part II, sign changes between positive to negative, by first derivative, a local maximum exist at $x=-2$ and the maximum value is $f(-2)=11$



13. **If $f(x)=2x^3+3x^2-36x$, find the intervals on which it is increasing or decreasing, the local maximum and local minimum values of f(x).**

Given $f(x)=2x^3+3x^2-36x$ and $f'(x)=6x^2+6x-36$

$f'(x)=0 \Rightarrow x^2+x-6=0 \Rightarrow (x-2)(x+3)=0 \Rightarrow x=2,-3$

Critical Point	Interval	Value of $f'(x)$
$x=2$ Part I $x=2$	$-3<x<2$ $x>2$	$f'(x)<0$ $f'(x)>0$
$x=-3$ Part II $x=-3$	$x<-3$ $-3<x<2$	$f'(x)>0$ $f'(x)<0$

In part I, sign changes between negative to positive, by first derivative, a local minimum exist at $x=2$ and the minimum value is $f(2)=16+12-72=-44$

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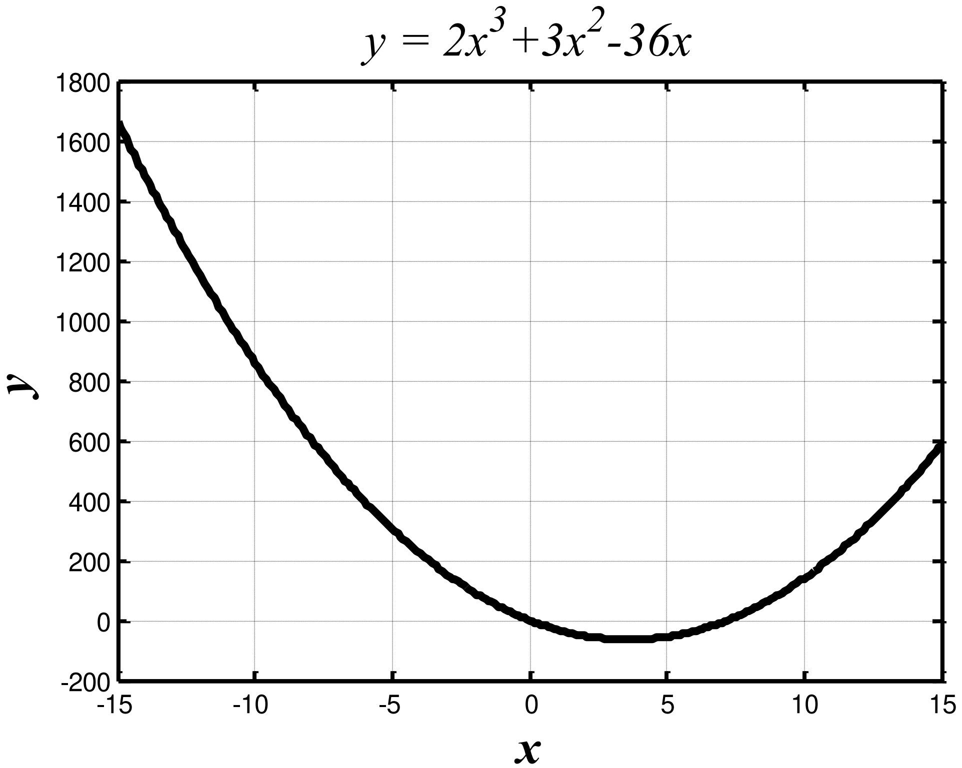
In part II, sign changes between positive to negative, by first derivative, a local maximum exist at $x=-3$ and the maximum value is $f(-3)=-54+27+108=81$



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UNIT III - FUNCTIONS OF SEVERAL VARIABLES

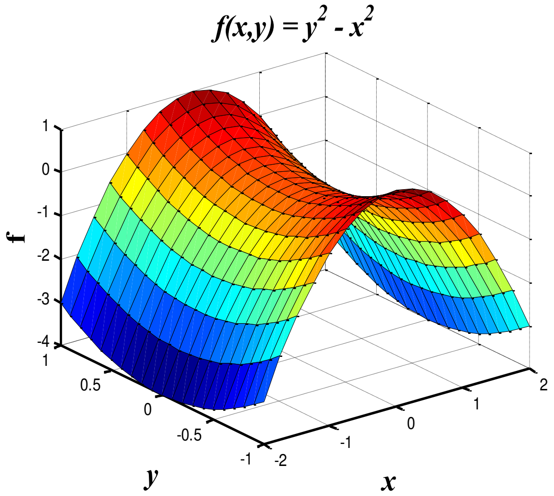
Introduction

A function of several variables consists of two parts; a domain, which is a collection of points in the plane or in space, and a rule, which assigns to each member of the domain one and only one number. If the domain is a set of points in the plane, it is called function of two variables. If the domain is a set of points in the sphere, it is called function of three variables.

The following are some functions of several variables along with its geometric interpretations:

1. $f(x,y)=xy,\ for\ x\geq 0\ \ and\ \ y\geq 0.$ - Area of a rectangle
2. $f(x,y,z)=xyz,\ for\ x\geq 0,\ y\geq 0\ \ and\ \ z\geq 0.$ - Volume of a rectangular parallelepiped

A computer drawn graph of $f(x,y)=y^2-x^2$ is given here.



Partial Derivatives

Let f be a function of two variables. If we fix one of the two variables, say $y=y_0$, the function whose values are $f(x,y_0)$ is a function of x alone. If f has a derivative at x_0, we call the derivative a partial derivative at (x_0,y_0).

Let f be a function of two variables and let (x_0,y_0) be in the domain of f. The partial derivatives of f with respect to x at (x_0,y_0) is defined by $f_x(x_0,y_0)=\underset{h\to 0}{Lt}\dfrac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$ provided that this limit exists. Similarly, the partial derivatives of f with respect to y at (x_0,y_0) is defined by $f_y(x_0,y_0)=\underset{h\to 0}{Lt}\dfrac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$ provided that this limit exists.

Also we can extend this concept to a functions of three variables also.

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The partial derivatives are denoted by $f_x = \frac{\partial f}{\partial x}$ *and* $f_y = \frac{\partial f}{\partial y}$.

Example: Let $f(x,y) = 2xy^2 - 3x^2y$. Find f_x *and* f_y and evaluate f_x *and* f_y at $(1,-1)$.

Keeping y as constant and differentiating f with respect to x, we find that $f_x(x,y) = 2y^2 - 6xy$
Therefore $f_x(1,-1) = 2+6 = 8$

Keeping x as constant and differentiating f with respect to y, we find that $f_y(x,y) = 4xy - 3x^2$
Therefore $f_y(1,-1) = -4-3 = -7$

The sum, product and quotient rules for derivatives are applicable for partial derivatives also. Thus f *and* g have partial derivatives, then

$$(f+g)_x = f_x + g_x \quad and \quad (f+g)_y = f_y + g_y$$
$$(f-g)_x = f_x - g_x \quad and \quad (f-g)_y = f_y - g_y$$
$$(f.g)_x = f_x.g + f.g_x \quad and \quad (f.g)_y = f_y.g + f.g_y$$
$$\left(\frac{f}{g}\right)_x = \frac{f_x.g - f.g_x}{g^2} \quad and \quad \left(\frac{f}{g}\right)_y = \frac{f_y.g - f.g_y}{g^2}$$

Geometrical Meaning

Suppose $f(x,y)$ is the temperature at any point (x,y) on a flat metal plate lying on the xy plane. Then $f_x(x_0,y_0)$ is the rate at which the temperature changes at (x_0,y_0) along the line through (x_0,y_0) parallel to the x axis (i.e. y is fixed). If the temperature increases as x increases, then $f_x(x_0,y_0) > 0$, whereas if the temperature decreases as x increases, then $f_x(x_0,y_0) < 0$. Similarly we can explain $f_y(x_0,y_0)$.

Thus $f_x(x_0,y_0)$ represents the slope of the tangent to the surface $f(x,y)$, parallel to the line $y = y_0$, at $(x_0,y_0,f(x_0,y_0))$. An analogous statement is true for $f_y(x_0,y_0)$ also.

Higher Order Partial Derivatives

For functions of three variables, partial derivatives at (x_0,y_0,z_0) are defined as follows:

$$f_x(x_0,y_0,z_0) = \underset{h\to 0}{Lt} \frac{f(x_0+h,y_0,z_0) - f(x_0,y_0,z_0)}{h}.$$

Similarly $f_y(x_0,y_0,z_0)$ and $f_y(x_0,y_0,z_0)$ can be defined.

A function of two variables may have second, third and higher derivatives. The higher order derivatives are denoted as follows:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{yy} = \frac{\partial^2 f}{\partial y^2}, f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, f_{yx} = \frac{\partial^2 f}{\partial y \partial x} \;\&\; f_{xxx} = \frac{\partial^3 f}{\partial x^3}, f_{yyy} = \frac{\partial^3 f}{\partial y^3}, f_{xxy} = \frac{\partial^3 f}{\partial x^2 \partial y}, f_{xyy} = \frac{\partial^3 f}{\partial x \partial x^2}$$

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Note: If f_{xy} & f_{yx} are continuous at $\left(x_0, y_0\right)$, then $f_{xy}\left(x_0, y_0\right) = f_{yx}\left(x_0, y_0\right)$

Solved Problems on Partial Differentiation

1. If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Given $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$. Differentiate partially w.r.t x, y, z respectively

$$\frac{\partial u}{\partial x} = -\frac{z}{x^2} + \frac{1}{y}, \quad \frac{\partial u}{\partial y} = -\frac{x}{y^2} + \frac{1}{z}, \quad \frac{\partial u}{\partial z} = -\frac{y}{z^2} + \frac{1}{x}.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{xz}{x^2} + \frac{x}{y} - \frac{yx}{y^2} + \frac{y}{z} - \frac{zy}{z^2} + \frac{z}{x} = -\frac{z}{x} + \frac{x}{y} - \frac{x}{y} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x} = 0.$$

2. Find $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$ *if* $u = y^x$

Given $u = y^x$

Taking log on both sides

$$\log u = \log y^x$$
$$\log u = x \log y$$
$$e^{\log u} = e^{x \log y}$$
$$u = e^{x \log y}$$

Partially differentiating with respect to x & y.

$\frac{\partial u}{\partial x} = e^{x \log y}(\log y)$ and $\frac{\partial u}{\partial y} = e^{x \log y}\left(\frac{x}{y}\right)$

3. Find $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$ *if* $u = x^y$

Given $u = x^y$

Taking log on both sides

$$\log u = \log x^y$$
$$\log u = y \log x$$
$$e^{\log u} = e^{y \log x}$$
$$u = e^{y \log x}$$

Partially differentiating with respect to x & y.

$\frac{\partial u}{\partial x} = e^{y \log x}\left[\frac{y}{x}\right]$ and $\frac{\partial u}{\partial y} = e^{y \log x}.\log x$

4. If $u = (x-y)(y-z)(z-x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Given $u = (x-y)(y-z)(z-x)....(1)$

Differentiate (1) partially w.r.t x,

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (1)(z-x)\right] = -(x-y)(y-z) + (y-z)(z-x)$$

Differentiate (1) partially w.r.t y,

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (-1)(y-z)\right] = (x-y)(z-x) - (y-z)(z-x)$$

Differentiate (1) partially w.r.t z,

$$\frac{\partial u}{\partial z} = (x-y)\left[(z-x)(-1) + (1)(y-z)\right] = -(x-y)(z-x) + (y-z)(x-y)$$

Adding, we get $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

5. **If** $u=\log(\tan x+\tan y+\tan z)$**, prove that** $\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2$

Given $u=\log(\tan x+\tan y+\tan z)$

$$\frac{\partial u}{\partial x}=\frac{\sec^2 x}{\tan x+\tan y+\tan z} \qquad \sin 2x\frac{\partial u}{\partial x}=\frac{2\sin x\cos x\sec^2 x}{\tan x+\tan y+\tan z}=\frac{2\tan x}{\tan x+\tan y+\tan z}$$

$$\frac{\partial u}{\partial y}=\frac{\sec^2 y}{\tan x+\tan y+\tan z} \qquad \sin 2y\frac{\partial u}{\partial y}=\frac{2\sin y\cos y\sec^2 y}{\tan x+\tan y+\tan z}=\frac{2\tan y}{\tan x+\tan y+\tan z}$$

$$\frac{\partial u}{\partial z}=\frac{\sec^2 z}{\tan x+\tan y+\tan z} \qquad \sin 2z\frac{\partial u}{\partial z}=\frac{2\sin z\cos z\sec^2 z}{\tan x+\tan y+\tan z}=\frac{2\tan z}{\tan x+\tan y+\tan z}$$

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=\frac{2\tan x+2\tan y+2\tan z}{\tan x+\tan y+\tan z}=2$$

6. **If** $u=\log(x^2+y^2+z^2)$**, prove that** $x\frac{\partial^2 u}{\partial y\partial z}=y\frac{\partial^2 u}{\partial x\partial z}=z\frac{\partial^2 u}{\partial x\partial y}$

Given $u=\log(x^2+y^2+z^2)$

$$\frac{\partial u}{\partial x}=\frac{2x}{\log(x^2+y^2+z^2)} \qquad \frac{\partial u}{\partial y}=\frac{2y}{\log(x^2+y^2+z^2)}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{-2x\frac{2y}{(x^2+y^2+z^2)}}{[\log(x^2+y^2+z^2)]^2} \qquad \frac{\partial^2 u}{\partial z\partial y}=\frac{-2y\frac{2z}{(x^2+y^2+z^2)}}{[\log(x^2+y^2+z^2)]^2}$$

$$z\frac{\partial^2 u}{\partial x\partial y}=\frac{-4xyz}{[\log(x^2+y^2+z^2)]^2(x^2+y^2+z^2)} \qquad x\frac{\partial^2 u}{\partial z\partial y}=\frac{-4xyz}{[\log(x^2+y^2+z^2)]^2(x^2+y^2+z^2)}$$

Similarly, we can prove $y\frac{\partial^2 u}{\partial z\partial x}=\frac{-4xyz}{[\log(x^2+y^2+z^2)]^2(x^2+y^2+z^2)}$

7. **If** $u=x^3+y^3-3axy$**, prove that** $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$

Given $u=x^3+y^3-3axy$

$$\frac{\partial u}{\partial x}=3x^2-3ay \qquad \frac{\partial u}{\partial y}=3y^2-3ax$$

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$$\frac{\partial^2 u}{\partial x\partial y} = -3a \qquad \frac{\partial^2 u}{\partial y\partial x} = -3a$$

8. If $u = e^{ax}\sin by$**, prove that** $\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\partial y}$

Given $u = e^{ax}\sin by$

$$\frac{\partial u}{\partial x} = ae^{ax}\sin by \qquad \frac{\partial u}{\partial y} = be^{ax}\cos by$$

$$\frac{\partial^2 u}{\partial y\partial x} = abe^{ax}\cos by \qquad \frac{\partial^2 u}{\partial x\partial y} = abe^{ax}\cos by$$

9. If $u = \tan^{-1}\frac{y}{x}$**, prove that** $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial u}{\partial x} = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2xy}{\left(x^2+y^2\right)^2} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{\left(x^2+y^2\right)^2}$$

10. If $u = \log\left(x^2+y^2\right)$**, prove that** $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial u}{\partial x} = \frac{2x}{\log\left(x^2+y^2\right)} \qquad \frac{\partial u}{\partial y} = \frac{2y}{\log\left(x^2+y^2\right)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2\log\left(x^2+y^2\right) - 2x\frac{2x}{\left(x^2+y^2\right)}}{\left[\log\left(x^2+y^2\right)\right]^2} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{2\log\left(x^2+y^2\right) - 2y\frac{2y}{\left(x^2+y^2\right)}}{\left[\log\left(x^2+y^2\right)\right]^2}$$

11. If $u = \log\left(x^3+y^3+z^3-3xyz\right)$ **show that** $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$

Given $u = \log\left(x^3+y^3+z^3-3xyz\right)$

Here $\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$; $\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$; $\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

Therefore

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3\left(x^2+y^2+z^2-xy-yz-xz\right)}{x^3+y^3+z^3-3xyz} = \frac{3\left(x^2+y^2+z^2-xy-yz-xz\right)}{(x+y+z)\left(x^2+y^2+z^2-xy-yz-xz\right)} = \frac{3}{x+y+z}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

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$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$=-\frac{9}{(x+y+z)^2}$$

12. If f is a function of $u\ and\ v$ and $u=e^x\cos y,\ v=e^x\sin y$, prove that $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\left(u^2+v^2\right)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)$

Since f is a function of $u\ and\ v$ and $u,\ v$ are functions of $x,\ y$, we have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=e^x\cos y\frac{\partial f}{\partial u}+e^x\sin y\frac{\partial f}{\partial v}=u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}$$

and hence $\frac{\partial}{\partial x}=u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}$

$$\frac{\partial^2 f}{\partial x^2}=\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}\right)$$

$$=u\frac{\partial}{\partial u}\left(u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}\right)+v\frac{\partial}{\partial v}\left(u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}\right)$$

$$=u\left(u\frac{\partial^2 f}{\partial u^2}+\frac{\partial f}{\partial u}\right)+uv\frac{\partial^2 f}{\partial u\partial v}+uv\frac{\partial^2 f}{\partial v\partial u}+v\left(v\frac{\partial^2 f}{\partial v^2}+\frac{\partial f}{\partial v}\right).....(1)$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=-e^x\sin y\frac{\partial f}{\partial u}+e^x\cos y\frac{\partial f}{\partial v}=-v\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial v}$$

and hence $\frac{\partial}{\partial y}=-v\frac{\partial}{\partial u}+u\frac{\partial}{\partial v}$

$$\frac{\partial^2 f}{\partial y^2}=\left(-v\frac{\partial}{\partial u}+u\frac{\partial}{\partial v}\right)\left(-v\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial v}\right)$$

$$=-v\frac{\partial}{\partial u}\left(-v\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial v}\right)+u\frac{\partial}{\partial v}\left(-v\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial v}\right)$$

$$=v^2\frac{\partial^2 f}{\partial u^2}-v\left(u\frac{\partial^2 f}{\partial u\partial v}+\frac{\partial f}{\partial v}\right)-u\left(v\frac{\partial^2 f}{\partial v\partial u}+\frac{\partial f}{\partial u}\right)+u^2\frac{\partial^2 f}{\partial v^2}.....(2)$$

Adding (1) and (2), we have

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$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\left(u^2\frac{\partial^2 f}{\partial u^2}+u\frac{\partial f}{\partial u}\right)+uv\frac{\partial^2 f}{\partial u\partial v}+uv\frac{\partial^2 f}{\partial v\partial u}+\left(v^2\frac{\partial^2 f}{\partial v^2}+v\frac{\partial f}{\partial v}\right)+v^2\frac{\partial^2 f}{\partial u^2}$$

$$-v\left(u\frac{\partial^2 f}{\partial u\partial v}+\frac{\partial f}{\partial v}\right)-u\left(v\frac{\partial^2 f}{\partial v\partial u}+\frac{\partial f}{\partial u}\right)+u^2\frac{\partial^2 f}{\partial v^2}$$

$$=\left(u^2+v^2\right)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)$$

13. If f is a function of u *and* v and $u=x^2-y^2,\ v=2xy$, prove that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4\left(x^2+y^2\right)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)$$

Since f is a function of u *and* v and $u,\ v$ are functions of $x,\ y$, we have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=2x\frac{\partial f}{\partial u}+2y\frac{\partial f}{\partial v}\ \text{ and hence }\ \frac{\partial}{\partial x}=2x\frac{\partial}{\partial u}+2y\frac{\partial}{\partial v}$$

$$\frac{\partial^2 f}{\partial x^2}=\left(2x\frac{\partial}{\partial u}+2y\frac{\partial}{\partial v}\right)\left(2x\frac{\partial f}{\partial u}+2y\frac{\partial f}{\partial v}\right)$$

$$=2x\frac{\partial}{\partial u}\left(2x\frac{\partial f}{\partial u}+2y\frac{\partial f}{\partial v}\right)+2y\frac{\partial}{\partial v}\left(2x\frac{\partial f}{\partial u}+2y\frac{\partial f}{\partial v}\right)$$

$$=4x^2\frac{\partial^2 f}{\partial u^2}+4xy\frac{\partial^2 f}{\partial u\partial v}+4xy\frac{\partial^2 f}{\partial v\partial u}+4y^2\frac{\partial^2 f}{\partial v^2}\text{..........(1)}$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=-2y\frac{\partial f}{\partial u}+2x\frac{\partial f}{\partial v}\ \text{ and hence }\ \frac{\partial}{\partial y}=-2y\frac{\partial}{\partial u}+2x\frac{\partial}{\partial v}$$

$$\frac{\partial^2 f}{\partial y^2}=\left(-2y\frac{\partial}{\partial u}+2x\frac{\partial}{\partial v}\right)\left(-2y\frac{\partial f}{\partial u}+2x\frac{\partial f}{\partial v}\right)$$

$$=-2y\frac{\partial}{\partial u}\left(-2y\frac{\partial f}{\partial u}+2x\frac{\partial f}{\partial v}\right)+2x\frac{\partial}{\partial v}\left(-2y\frac{\partial f}{\partial u}+2x\frac{\partial f}{\partial v}\right)$$

$$=4y^2\frac{\partial^2 f}{\partial u^2}-4xy\frac{\partial^2 f}{\partial u\partial v}-4xy\frac{\partial^2 f}{\partial v\partial u}+4x^2\frac{\partial^2 f}{\partial v^2}\text{..........(2)}$$

Adding (1) and (2), we have

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4x^2\frac{\partial^2 f}{\partial u^2}+4xy\frac{\partial^2 f}{\partial u\partial v}+4xy\frac{\partial^2 f}{\partial v\partial u}+4y^2\frac{\partial^2 f}{\partial v^2}$$

$$+4y^2\frac{\partial^2 f}{\partial u^2}-4xy\frac{\partial^2 f}{\partial u\partial v}-4xy\frac{\partial^2 f}{\partial v\partial u}+4x^2\frac{\partial^2 f}{\partial v^2}$$

$$=4\left(x^2+y^2\right)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)$$

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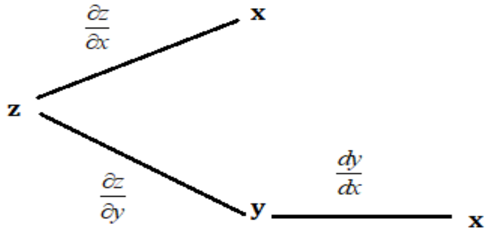
Implicit Differentiation

Suppose that $z = f(x, y)\ and\ \ y = g(x).$

Find a formula for $\dfrac{dz}{dx}$.

The diagram is



It follows that $\dfrac{dz}{dx} = \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dx}$ …….(1)

Consider the function $f(x, y) = 0\ where\ \ y = g(x).$ Then $f(x,\ g(x)) = 0.$ If $z = f(x, y),$ then by assumption $\dfrac{dz}{dx} = \dfrac{d}{dx} f(x, g(x)) = \dfrac{d}{dx}(0) = 0$

Therefore from (1),

$$0 = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$\frac{\partial z}{\partial y}\frac{dy}{dx} = -\frac{\partial z}{\partial x}$$

$$\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$$

Example: Let $x^3 + y^3 = 2xy.\ Find\ \dfrac{dy}{dx}.$

Let $z = x^3 + y^3 - 2xy$. Then

$$\frac{\partial z}{\partial x} = 3x^2 - 2y \quad and \quad \frac{\partial z}{\partial y} = 3y^2 - 2x$$

Then $\dfrac{dy}{dx} = -\dfrac{z_x}{z_y} = -\dfrac{3x^2 - 2y}{3y^2 - 2x}$

Example: Let $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2.\ Find\ \dfrac{dy}{dx}.$

Let $z = x^{\frac{2}{3}} + y^{\frac{2}{3}} - 2$. Then

$$\frac{\partial z}{\partial x} = \frac{2}{3}x^{-\frac{1}{3}} \quad and \quad \frac{\partial z}{\partial y} = \frac{2}{3}y^{-\frac{1}{3}}$$

Then $\dfrac{dy}{dx} = -\dfrac{z_x}{z_y} = -\dfrac{\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}} = -\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

Example: Let $e^{\frac{x}{y}} + \ln\dfrac{y}{x} + 15 = 0.\ \ Find\ \dfrac{dy}{dx}.$

Let $z = e^{\frac{x}{y}} + \ln\dfrac{y}{x} + 15.$

Then $\dfrac{\partial z}{\partial x} = e^{\frac{x}{y}}.\dfrac{1}{y} + \dfrac{x}{y}\left(-\dfrac{y}{x^2}\right)\ \ and\ \ \dfrac{\partial z}{\partial y} = e^{\frac{x}{y}}\left(-\dfrac{x}{y^2}\right) + \dfrac{x}{y}\left(\dfrac{1}{x}\right)$

$$= e^{\frac{x}{y}}.\frac{1}{y} - \frac{1}{x} \qquad\qquad = -\frac{x}{y^2}e^{\frac{x}{y}} + \frac{1}{y}$$

Then $\frac{dy}{dx} = -\frac{z_x}{z_y} = \dfrac{e^{\frac{x}{y}}.\frac{1}{y} - \frac{1}{x}}{\frac{x}{y^2}e^{\frac{x}{y}} + \frac{1}{y}}$

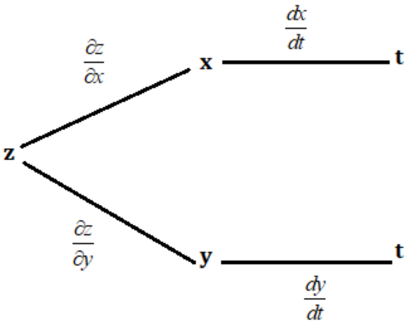
Total Derivative

If $z = f(x, y)$, where $x = g_1(t)$ *and* $y = g_2(t)$ then we can express z as a function of t alone by substituting the values of x *and* y in $f(x, y)$. Thus we can find the ordinary derivative $\frac{dz}{dt}$ which is called the total derivative of z. Here $dz = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy$

To evaluate $\frac{dz}{dt}$ without substituting the values of x *and* y in $f(x, y)$, we introduce the chain rule.

The Chain Rule

1. If $z = f(x, y)$, $x = g_1(t)$ & $y = g_2(t)$. Then $\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$



2. If $w = f(x, y, z)$, $x = g_1(t)$, $y = g_2(t)$ *and* $z = g_3(t)$ Then $\frac{dw}{dt} = \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} + \frac{\partial w}{\partial z}.\frac{dz}{dt}$

3. If $z = f(x, y)$, $x = g_1(u, v)$ & $y = g_2(u, v)$. Then $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$ & $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$

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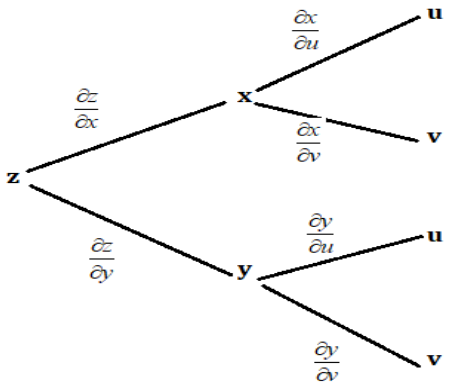

Solved Problems on total derivative

1. **Let** $z = x^2 e^y,\ x = \sin t\ and\ y = t^3.$ **Find** $\frac{dz}{dt}$**. Verify the result by direct substitution.**

We know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

$$= 2xe^y \cos t + x^2 e^y 3t^2$$

$$= 2(\sin t)e^{(t^3)} \cos t + 3\left(\sin^2 t\right)e^{(t^3)}t^2$$

By direct substitution: Given $z = x^2 e^y = \sin^2 t.e^{t^3}$

Now $\frac{dz}{dt} = \left(\sin^2 t\right).\left(3t^2 e^{t^3}\right) + \left(e^{t^3}\right)\left(2\sin t \cos t\right)$

2. **If** $z = x^2 + y^2$ **and** $x = t^2,\ y = 2at$**, find** $\frac{dz}{dt}$**.**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

$$= 2x.2t\ +\ 2y.2a$$

$$= 4xt + 4ay$$

3. **Let** $z = \sin\left(\frac{x}{y}\right),\ x = e^t\ and\ y = t^2.$ **Find** $\frac{dz}{dt}$**. Verify the result by direct substitution.**

We know that	By direct substitution:

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$$\frac{dz}{dt}=\frac{\partial z}{\partial x}.\frac{dx}{dt}+\frac{\partial z}{\partial y}.\frac{dy}{dt}$$

$$=\frac{1}{y}\cos\left(\frac{x}{y}\right).e^t-2t\left(\frac{x}{y^2}\right)\cos\left(\frac{x}{y}\right)$$

$$=\frac{1}{t^2}\cos\left(\frac{e^t}{t^2}\right).e^t-2t\left(\frac{e^t}{t^4}\right)\cos\left(\frac{e^t}{t^2}\right)$$

$$=\cos\left(\frac{e^t}{t^2}\right)\left[\frac{e^t}{t^2}-\frac{2te^t}{t^4}\right]$$

Given $z=x\sin\left(\frac{x}{y}\right)=\sin\left(\frac{e^t}{t^2}\right)$

Now $\frac{dz}{dt}=\cos\left(\frac{e^t}{t^2}\right)\left[\frac{t^2e^t-2te^t}{t^4}\right]$

4. **Let** $z=x\ln y,\ where\ x=u^2+v^2\ \ and\ \ y=u^2-v^2.$ **Find** $\frac{\partial z}{\partial u}$ **and** $\frac{\partial z}{\partial v}$**.**

We know that

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u}=(\ln y)(2u)+\left(\frac{x}{y}\right)(2u)$$

$$=2u.\ln\left(u^2-v^2\right)+2u.\frac{u^2+v^2}{u^2-v^2}$$

Also $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$

$$\frac{\partial z}{\partial v}=(\ln y)(2v)+\left(\frac{x}{y}\right)(-2v)$$

$$=2v.\ln\left(u^2-v^2\right)-2v.\frac{u^2+v^2}{u^2-v^2}$$

5. **Let** $w=x\cos yz^2,\ where\ x=\sin t,\ y=t^2\ and\ z=e^t.$ **Find** $\frac{dw}{dt}$

We know that $\frac{dw}{dt}=\frac{\partial w}{\partial x}.\frac{dx}{dt}+\frac{\partial w}{\partial y}.\frac{dy}{dt}+\frac{\partial w}{\partial z}.\frac{dz}{dt}$

$$\frac{dw}{dt}=\cos yz^2.\cos t-xz^2\sin yz^2(2t)-2xyz\sin yz^2\left(e^t\right)$$

$$=\cos\left(t^2e^{2t}\right).\cos t-2te^{2t}\sin\left(t^2e^{2t}\right)(2t)-2t^2e^{2t}\sin t\sin\left(t^2e^{2t}\right)$$

6. **Let** $w=\sqrt{x}+y^2z^3,\ where\ x=1+u^2+v^2,\ y=uv\ and\ z=3u.$ **Find** $\frac{\partial w}{\partial u}$ & $\frac{\partial w}{\partial v}$

We know that

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}.\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}.\frac{\partial z}{\partial u}$$

Also

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}.\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}.\frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial u}=\frac{1}{2\sqrt{x}}(2u)+2yz^3(v)+3y^2z^3(3)$$
$$=\frac{u}{\sqrt{1+u^2+v^2}}+54u^4v^2++81u^4v^2$$
$$=\frac{u}{\sqrt{1+u^2+v^2}}+135u^4v^2$$

$$\frac{\partial w}{\partial v}=\frac{1}{2\sqrt{x}}(2v)+2yz^3(u)+3y^2z^2(0)$$
$$=\frac{v}{\sqrt{1+u^2+v^2}}+2(uv)(3u)^3(u)$$
$$=\frac{v}{\sqrt{1+u^2+v^2}}+54u^5v$$

7. **If** $u=x\log xy$ **where** $x^3+y^3+3xy=1$**, find** $\frac{du}{dx}$.

Let $f(x,y)=x^3+y^3+3xy-1$. Then $\frac{dy}{dx}=-\frac{f_x}{f_y}$.

$f_x=3x^2+3y \quad and \quad f_y=y^2+3x$

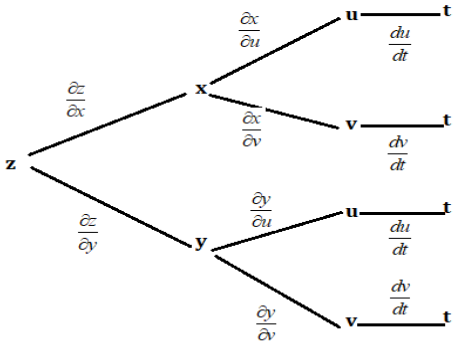
$$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{3x^2+3y}{3y^2+3x}=-\frac{x^2+y}{y^2+x}$$

Also

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}$$
$$=\left[x.\frac{1}{xy}.y+\log xy\right]+\left[x.\frac{1}{xy}.x+(0)\log xy\right]\left[-\frac{x^2+y}{y^2+x}\right]$$
$$=[1+\log xy]+\left[\frac{x}{y}\right]\left[-\frac{x^2+y}{y^2+x}\right]$$
$$=[1+\log xy]-\frac{x\left(x^2+y\right)}{y\left(y^2+x\right)}$$

8. **Suppose that** $z=f(x,y),\ x=g(u,v),\ y=h(u,v),\ u=j(t),\ v=k(t).$ **Find a formula for** $\frac{dw}{dt}$**.**

We draw the diagram



Using the four paths, leading from $w\ to\ t$, we find that

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{dt}$$

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Change of Variables

If $z = f(x,y)$ where $x = g_1(u,v)$ & $y = g_2(u,v)$. Sometimes it is necessary to change the expressions involving z, x, y, z_x, z_y *etc.* to expressions involving z, u, v, z_u, z_v *etc.*

If v is considered as a constant, then $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$. Similarly regarding u as constant, we have $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$. These are system of simultaneous equations in $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
On solving these equations, we get their values in terms of $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, z, u, v$.

Note: In the above, instead of the substitutions $x = g_1(u,v)$ & $y = g_2(u,v)$, suppose $u = \phi_1(x,y)$ & $v = \phi_2(x,y)$, then $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}$ & $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$

Solved Problems on Change of Variables

1. If $z = f(u-v, v-u)$, then show that $\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = 0$.

Let $r = u - v,\ s = v - u$ and hence z is a function of r, s and r, s are functions of u, v.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial z}{\partial s}\frac{\partial s}{\partial u} = \frac{\partial z}{\partial r}(1) + \frac{\partial z}{\partial s}(-1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial v} + \frac{\partial z}{\partial s}\frac{\partial s}{\partial v} = \frac{\partial z}{\partial r}(-1) + \frac{\partial z}{\partial s}(1)$$

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{\partial z}{\partial r}(1) + \frac{\partial z}{\partial s}(-1) + \frac{\partial z}{\partial r}(-1) + \frac{\partial z}{\partial s}(1) = 0$$

2. If $u = f(x-y, y-z, z-x)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Let $r = x - y,\ s = y - z,\ t = z - x$ and hence u is a function of r, s, t and r, s, t are functions of x, y, z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0$$

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3. **If** $z = f(x,y)$ **and** $x = e^{u} + e^{-v}$**,** $y = e^{-u} - e^{v}$, **prove that** $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

From the given data, $\frac{\partial x}{\partial u} = e^{u}$, $\frac{\partial x}{\partial v} = -e^{-v}$, $\frac{\partial y}{\partial u} = -e^{-u}$, $\frac{\partial y}{\partial v} = -e^{v}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}.e^{u} - \frac{\partial z}{\partial y}.e^{-u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = -\frac{\partial z}{\partial x}e^{-v} - \frac{\partial z}{\partial y}e^{v}$$

Subtracting

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.e^{u} - \frac{\partial z}{\partial y}.e^{-u} + \frac{\partial z}{\partial x}e^{-v} + \frac{\partial z}{\partial y}e^{v}$$

$$= \left(e^{u} + e^{-v}\right)\frac{\partial z}{\partial x} - \left(e^{-u} - e^{v}\right)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

4. **If** $z = f(x,y)$ **and** $x = e^{u}\cos v$**,** $y = e^{u}\sin v$ **prove that** $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$

From the given data, $\frac{\partial x}{\partial u} = e^{u}\cos v$, $\frac{\partial x}{\partial v} = -e^{u}\sin v$, $\frac{\partial y}{\partial u} = e^{u}\sin v$, $\frac{\partial y}{\partial v} = e^{u}\cos v$

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$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}.e^u\cos v+\frac{\partial z}{\partial y}.e^u\sin v$$

$$y\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}.e^u\cos v.e^u\sin v+\frac{\partial z}{\partial y}.e^u\sin v.e^u\sin v$$

$$=\frac{\partial z}{\partial x}\;e^{2u}\cos v.\sin v+\frac{\partial z}{\partial y}\;e^{2u}\sin^2 v............(1)$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}=-\frac{\partial z}{\partial x}e^u\sin v+\frac{\partial z}{\partial y}e^u\cos v$$

$$x\frac{\partial z}{\partial v}=-\frac{\partial z}{\partial x}e^u\sin v.e^u\cos v+\frac{\partial z}{\partial y}e^u\cos v.e^u\cos v$$

$$=-\frac{\partial z}{\partial x}e^{2u}\sin v\cos v+\frac{\partial z}{\partial y}e^{2u}\cos^2 v..........(2)$$

Adding (1) *and* (2)

$$x\frac{\partial z}{\partial v}+y\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\;e^{2u}\cos v.\sin v+\frac{\partial z}{\partial y}\;e^{2u}\sin^2 v-\frac{\partial z}{\partial x}e^{2u}\sin v\cos v+\frac{\partial z}{\partial y}e^{2u}\cos^2 v$$

$$=\frac{\partial z}{\partial y}\;e^{2u}\sin^2 v+\frac{\partial z}{\partial y}e^{2u}\cos^2 v$$

$$=\frac{\partial z}{\partial y}\;e^{2u}\left(\sin^2 v+\cos^2 v\right)$$

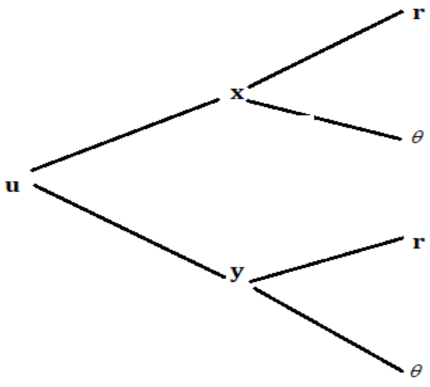
$$=\frac{\partial z}{\partial y}\;e^{2u}$$

5. Transform the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ into polar coordinates.

We know that the Cartesian and polar relationships are

$x=r\cos\theta,\ y=r\sin\theta\ \ and$

$r=\sqrt{x^2+y^2},\ \theta=\tan^{-1}\frac{y}{x}$



Now $\frac{\partial r}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{r}=\cos\theta$

Also $\frac{\partial\theta}{\partial x}=-\frac{1}{1+\frac{y^2}{x^2}}\frac{y}{x^2}=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r}$

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$$\frac{\partial r}{\partial y}=\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y}{r}=\sin\theta \quad \text{and} \quad \frac{\partial \theta}{\partial y}=-\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}=-\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}$$

Therefore $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}.\frac{\partial \theta}{\partial x}=\frac{\partial u}{\partial r}.\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$ and

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}.\frac{\partial \theta}{\partial y}=\frac{\partial u}{\partial r}.\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$

i.e. $\frac{\partial}{\partial x}=\frac{\partial}{\partial r}.\cos\theta-\frac{\partial}{\partial \theta}\frac{\sin\theta}{r}$ and $\frac{\partial}{\partial y}=\frac{\partial}{\partial r}.\sin\theta+\frac{\partial}{\partial \theta}\frac{\cos\theta}{r}$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right]=\left(\frac{\partial}{\partial r}.\cos\theta-\frac{\partial}{\partial \theta}\frac{\sin\theta}{r}\right)\left(\frac{\partial u}{\partial r}.\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)$$

$$=\frac{\partial^2 u}{\partial r^2}\cos^2\theta-2\frac{\partial^2 u}{\partial\theta\partial r}\frac{\sin\theta}{r}\cos\theta+\frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}+2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right]=\left(\frac{\partial}{\partial r}.\sin\theta+\frac{\partial}{\partial \theta}\frac{\cos\theta}{r}\right)\left(\frac{\partial u}{\partial r}.\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)$$

$$=\frac{\partial^2 u}{\partial r^2}\sin^2\theta+2\frac{\partial^2 u}{\partial\theta\partial r}\frac{\cos\theta}{r}\sin\theta+\frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}-2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

Adding, we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}$$

Exercise

1 Use partial derivative to find $\frac{d^2y}{dx^2}$, when $x^4+y^4=4a^2xy$

2 Use partial derivative to find $\frac{d^2y}{dx^2}$, when $x^3+y^3=3axy$

3 If $u=x^y$, show that $\frac{\partial^3 u}{\partial x^2\partial y}=\frac{\partial^3 u}{\partial x\partial y\partial x}$

4 Find $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$ if (i) $z=x^2y-x\sin xy$ (ii) $z(x+y)=x^2+y^2$

5 Verify $f_{xy}=f_{yx}$ if $f=\sin^{-1}\frac{y}{x}$

Chain Rule

1 If $u=xy+yz+xz$, where $x=e^t$, $y=e^{-t}$ and $z=\frac{1}{t}$, find $\frac{du}{dt}$.

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2 If $u=\sin^{-1}(x-y)$, where $x=3t,\ y=4t^3$ and show that $\frac{du}{dt}=\frac{3}{\sqrt{1-t^2}}$

3 If $u=f(x,y)$, where $x=r\cos\theta,\ y=r\sin\theta$, prove that $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

4 If $u=x^3y^2+y^3x^2$, where $x=at^2,\ y=2at$, find $\frac{du}{dt}$.

5 If $z=f(u,v)$, where $u=x^2-y^2,\ v=2xy$ prove that

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4\left(x^2+y^2\right)\left(\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right)$$

Substitution Method

1 If $u=f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x},\right)$ prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$.

2 If $z=f\left(cx-az,cy-bz\right)$ prove that $a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c$.

3 If $f\left(\frac{y-x}{xy},\frac{z-x}{zx}\right)=0$, show that $x^2\frac{\partial f}{\partial x}+y^2\frac{\partial f}{\partial y}=0$.

4 If $z=f\left(u,v\right)$, where $u=lx+my,\ v=ly-mx$, show that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(l^2+m^2\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$.

Implicit Function

1 Find $\frac{dy}{dx}$, if $x^3+y^3=3ax^2y$

2 Find $\frac{dy}{dx}$, if $x^y+y^x=c$

3 Find $\frac{du}{dx}$, if $u=\left(x^2+y^2\right)$, where $a^2x^2+b^2y^2=c^2$

4 Find $\frac{du}{dx}$, if $u=\tan^{-1}\frac{y}{x}$, where $x^2+y^2=c^2$

Jacobians

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If $u \ and \ v$ are functions of two independent variables $x \ and \ y$, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and is denoted by $\frac{\partial(u,v)}{\partial(x,y)} \ or \ J\left(\frac{u,v}{x,y}\right)$.

Similarly we can extend this concept to any number of variables.

Write the formula for Jacobian of u, v, w with respect to x, y, z.

The Jacobian of u, v, w with respect to x, y, z is $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Properties of Jacobians

1. **Prove that** $\frac{\partial(u,v)}{\partial(x,y)}.\frac{\partial(x,y)}{\partial(u,v)} = 1.$

Let $u = f(x,y) \ and \ v = g(x,y)$. Rewriting this as $x = \phi(u,v) \ and \ y = \psi(u,v)$.

Then $\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial u} \quad and \quad \frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial v}$

$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial u} \quad and \quad \frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial v}$

$$\frac{\partial(u,v)}{\partial(x,y)}.\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial u} \\ \frac{\partial u}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial v} & \frac{\partial v}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 1$$

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Note: If $J=\frac{\partial(u,v)}{\partial(x,y)}, then\ J'=\frac{\partial(x,y)}{\partial(u,v)}. Therefore\ JJ'=1$

2. Chain Rule: If u , v are functions of r , s and r , s are functions of $x\ and\ y$, then

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}.\frac{\partial(r,s)}{\partial(x,y)}$$

Consider

$$\frac{\partial(u,v)}{\partial(r,s)}.\frac{\partial(r,s)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial s}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s}\end{vmatrix}\times\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial s}{\partial x}\\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}\\ \frac{\partial v}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial v}{\partial s}.\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial v}{\partial s}.\frac{\partial s}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$=\frac{\partial(u,v)}{\partial(x,y)}$$

3. If u , v , w are said to be functionally dependent functions (i.e. there exists a relationship among them) of independent variables x , y , z then $\frac{\partial(u,v,w)}{\partial(x,y,z)}=0$ and vice versa.

Note: Jacobians is used to change the variables in multiple integrals. If the transformations $x=x(u,v)\ and\ y=y(u,v)$ are made in the double integral $\iint f(x,y)dxdy$, then $f(x,y)=F(u,v)$ and $dxdy=|J|dudv$, where $J=\frac{\partial(x,y)}{\partial(u,v)}$. This can be extended to triple integral also

Solved Problems on Jacobians

1. Find $\frac{\partial(x,y)}{\partial(r,\theta)}$, if $x=rcos\theta$ and $y=rsin\theta$

$x=rcos\theta$, $y=rsin\theta$

$\frac{\partial x}{\partial r}=cos\theta$, $\frac{\partial y}{\partial r}=\sin\theta$

$\frac{\partial x}{\partial\theta}=-rsin\theta$, $\frac{\partial y}{\partial\theta}=r\cos\theta$

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$$\frac{\partial(\mathrm{x},\mathrm{y})}{\partial(\mathrm{r},\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\left[\cos^2\theta+\sin^2\theta\right]=r$$

2. **If** $x=r\cos\theta,\ y=r\sin\,\theta$ **find** $\frac{\partial(r,\theta)}{\partial(x,y)}$.

We know from the previous example that $\frac{\partial(\mathrm{x},\mathrm{y})}{\partial(\mathrm{r},\theta)}=r$

By a property $\frac{\partial(\mathrm{x},\mathrm{y})}{\partial(\mathrm{r},\theta)}\frac{\partial(\mathrm{r},\theta)}{\partial(\mathrm{x},\mathrm{y})}=1$

$$(\mathrm{r})\frac{\partial(\mathrm{r},\theta)}{\partial(\mathrm{x},\mathrm{y})}=1$$

$$\frac{\partial(\mathrm{r},\theta)}{\partial(\mathrm{x},\mathrm{y})}=\frac{1}{r}$$

3. **If** $u=\frac{y^2}{2x},v=\frac{x^2+y^2}{2x}$**,** **find** $\frac{\partial(u,v)}{\partial(x,y)}$.

$$u=\frac{y^2}{2x},\qquad v=\frac{x^2+y^2}{2x}$$

$$\frac{\partial u}{\partial x}=\frac{-y^2}{2x^2},\qquad \frac{\partial v}{\partial x}=\frac{2x(2x)-(x^2+y^2)2}{(2x)^2}=2\frac{2x^2-x^2-y^2}{4x^2}=\frac{x^2-y^2}{2x^2}$$

$$\frac{\partial u}{\partial y}=\frac{y}{x},\qquad \frac{\partial v}{\partial y}=\frac{y}{x}$$

$$\frac{\partial(\mathrm{u},\mathrm{v})}{\partial(\mathrm{x},\mathrm{y})}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{-y^2}{2x^2} & \frac{y}{x}\\ \frac{x^2-y^2}{2x} & \frac{y}{x}\end{vmatrix}$$

$$=\left[-\frac{y^2}{2x^2}\frac{y}{x}-\frac{y}{x}\frac{x^2-y^2}{2x}\right]$$

$$=\frac{y}{x}\left[-\frac{y^2}{2x^2}-\frac{x^2-y^2}{2x}\right]$$

$$=\frac{y}{x}\left[-\frac{y^2}{2x^2}-\frac{x}{2}+\frac{y^2}{2x}\right]$$

4. If $x=u^2-v^2,\ y=2uv$ evaluate the Jacobian of $x,\ y$ with respect to u, v.

Given : $x = u^2 - v^2$ $\qquad y = 2uv$

$$\frac{\partial x}{\partial u} = 2u \qquad \frac{\partial x}{\partial u} = 2v$$

$$\frac{\partial x}{\partial v} = -2v \qquad \frac{\partial x}{\partial v} = 2u$$

$$\frac{\partial(\mathrm{x},\mathrm{y})}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= 4u^2 + 4v^2$$

5. **If** $u = xy$ and $v = x + y$ **find** $\frac{\partial(x,y)}{\partial(u,v)}$

Given : $u = xy$ $\qquad v = x + y$

$$\frac{\partial u}{\partial x} = y \qquad \frac{\partial v}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = x \qquad \frac{\partial v}{\partial y} = 1$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 1 \end{vmatrix} = y - x$$

$$Hence\ \frac{\partial(x,y)}{\partial(u,u)} = \frac{1}{y-x}$$

6. **If** $u = \frac{y^2}{x},\ v = \frac{x^2}{y}$**, find** $\frac{\partial(x,y)}{\partial(u,v)}$**.**

Given : $u = \frac{y^2}{x}$ $\qquad v = \frac{x^2}{y}$

$$\frac{\partial u}{\partial x} = -\frac{y^2}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{2x}{y}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x} \qquad \frac{\partial v}{\partial y} = -\frac{x^2}{y^2}$$

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$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & -\frac{x^2}{y^2} \end{vmatrix}$$

$$= 1 - 4$$

$$= -3$$

$$Hence \;\; \frac{\partial(x,y)}{\partial(u,u)} = -\frac{1}{3}$$

7. If $x = u(1-v)$**,** $y = uv$**, Prove that** $\frac{\partial(u,v)}{\partial(x,y)} . \frac{\partial(x,y)}{\partial(u,v)} = 1.$

Given : $x = u(1-v) \quad y = uv$

$$\therefore \frac{\partial x}{\partial u} = 1 - v \qquad \therefore \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = -u \qquad \frac{\partial y}{\partial v} = u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u(1-v) + uv$$

$$= u - uv + uv = u$$

Rewrite u & v in terms of x & y.

$$x = u(1-v) = u - uv \qquad y = uv$$

$$x = u - y \qquad v = \frac{y}{u} = \frac{y}{x+y}$$

$$\therefore \; u = x + y \text{ and } v = \frac{y}{x+y}$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial y} = \left[\frac{(x+y)-y}{(x+y)^2}\right] = \left[\frac{x}{(x+y)^2}\right]$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{x+y}{(x+y)^2}$$

$$= \frac{1}{(x+y)}$$

$$= \frac{1}{u}$$

$$\therefore \; \frac{\partial(u,v)}{\partial(x,y)} . \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{u} \, u = 1$$

8. **If $x = u\cos v$ and $y = u\sin v$, prove that $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$**

Given $x = u\cos v$ and $y = u\sin v$

$$\therefore \frac{\partial x}{\partial u} = \cos v \qquad \therefore \frac{\partial y}{\partial u} = \sin v$$

$$\frac{\partial x}{\partial v} = -u\sin v \qquad \therefore \frac{\partial y}{\partial v} = u\cos v$$

Also $\frac{\sin v}{\cos v} = \frac{y}{x}$ *i.e.* $\tan v = \frac{y}{x}$ and $u^2\cos^2 v + u^2\sin^2 v = x^2 + y^2$ *i.e.* $u^2 = x^2 + y^2$

i.e. $v = \tan^{-1}\frac{y}{x}$ and $u = \sqrt{x^2+y^2}$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}.\frac{-y}{x^2} = -\frac{x^2}{x^2+y^2}.\frac{y}{x^2} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}.\frac{1}{x} = \frac{x^2}{x^2+y^2}.\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{\left(x^2+y^2\right)\sqrt{x^2+y^2}} + \frac{y^2}{\left(x^2+y^2\right)\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2}{\left(x^2+y^2\right)\sqrt{x^2+y^2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix}$$

$$= u\cos^2 v + u\sin^2 v$$

$$= u$$

Therefore $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\sqrt{x^2+y^2}}.u = \frac{u}{u} = 1$

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9. If $x = v^2 + w^2, y = w^2 + u^2, z = u^2 + v^2$, find J'.

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\frac{\partial x}{\partial u} = 0 \qquad \frac{\partial x}{\partial v} = 2v \qquad \frac{\partial x}{\partial w} = 2w$$

$$\frac{\partial y}{\partial u} = 2u \qquad \frac{\partial y}{\partial v} = 0 \qquad \frac{\partial y}{\partial w} = 2w$$

$$\frac{\partial z}{\partial u} = 2u \qquad \frac{\partial z}{\partial v} = 2v \qquad \frac{\partial z}{\partial w} = 0$$

$$\mathrm{J} = \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix} = 0 - 2v(0 - 4uw) + 2w(4uv - 0) = 8uvw + 8uvw = 16uvw$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = 16uvw$$

$$J' = \frac{1}{J} = \frac{1}{16uvw}$$

10. If $x = r\cos\theta,\ y = r\sin\theta, z = z$. Find the Jacobian of x, y, z in terms of r, θ, z.

$$Given:\quad x = r\cos\theta,\ y = r\sin\theta, z = z$$

$$\mathrm{J} = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial x}{\partial z} = 0 \qquad \frac{\partial z}{\partial r} = 0 \qquad \frac{\partial z}{\partial \theta} = 0$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta \qquad \frac{\partial y}{\partial z} = 0 \qquad \frac{\partial z}{\partial z} = 1$$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 - 0 + 1(\cos\theta(r\cos\theta) - \sin\theta(-r\sin\theta))$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r(1) = r$$

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11. If $x = u(1+v), \quad y = v(1+u)$ **find** $\dfrac{\partial(x,y)}{\partial(u,v)}$**.**

Given : $x = u(1+v) \qquad\qquad y = v(1+u)$

$$\therefore \frac{\partial x}{\partial u} = 1+v \qquad\qquad \therefore \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = u \qquad\qquad \frac{\partial y}{\partial v} = 1+u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+u)(1+v) - uv$$

$$= 1+u+v+uv-uv = 1+u+v$$

12. If $x = uv, \quad y = \dfrac{u}{v}$ **show that** $J.J' = 1$

$$Given : x = uv \qquad\qquad y = \frac{u}{v}$$

$$\therefore xy = u^2 \qquad\qquad \therefore u = \sqrt{xy}$$

$$\frac{y}{x} = \frac{1}{v^2} \qquad\qquad v = \sqrt{\frac{x}{y}}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \dfrac{1}{v} & \dfrac{-u}{v^2} \end{vmatrix} = -2\frac{u}{v} \quad ...(1)$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{\sqrt{y}}{2\sqrt{x}} & \dfrac{\sqrt{x}}{2\sqrt{y}} \\ \dfrac{1}{2\sqrt{xy}} & -\dfrac{1}{2}\dfrac{\sqrt{x}}{\sqrt{y}}.\dfrac{1}{y} \end{vmatrix} = -\frac{\sqrt{xy}}{4y\sqrt{xy}} - \frac{\sqrt{x}}{4y\sqrt{x}} = -\frac{1}{2y}$$

$$J.J' = \left(-\frac{1}{2y}\right)\left(\frac{-2u}{v}\right) = \frac{u}{yv} = 1 \;\; \sin ce \; y = \frac{u}{v}$$

13. If $x+y+z=u$**,** $y+z=uv$**,** $z=uvw$**. Prove that** $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

$$x+y+z=u \qquad\qquad y+z=uv \qquad\qquad z=uvw$$

$$x+uv=u \qquad\qquad y+uvw=uv$$

$$x=u-uv \qquad\qquad y=uv-uvw$$

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x	y	z
$x = u(1-v)$	$x = uv(1-w)$	$z = uvw$
$\frac{\partial x}{\partial u} = 1-v$	$\frac{\partial y}{\partial u} = v(1-w)$	$\frac{\partial z}{\partial u} = vw$
$\frac{\partial x}{\partial v} = -u$	$\frac{\partial y}{\partial v} = u(1-w)$	$\frac{\partial z}{\partial v} = uw$
$\frac{\partial x}{\partial w} = 0$	$\frac{\partial y}{\partial w} = -uv$	$\frac{\partial z}{\partial w} = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

Taking u from 2nd and 3rd column & v from 3rd column

$$= u^2v \begin{vmatrix} 1-v & -1 & 0 \\ v(1-w) & (1-w) & -1 \\ vw & w & 1 \end{vmatrix}$$

$$= u^2v\left[(1-v)(1-w+w) + (v-vw+vw)\right]$$

$$= u^2v\left[(1-v)+v\right]$$

14. When $u = \frac{yz}{x}$ and $v = \frac{xz}{y}$, determine $\frac{\partial(u,v)}{\partial(x,y)}$

Given $u = \frac{yz}{x}$ and $v = \frac{xz}{y}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} \\ \frac{z}{y} & -\frac{xz}{y^2} \end{vmatrix} = \frac{xyz^2}{x^2y^2} - \frac{z^2}{xy} = \frac{z^2}{xy}(1-1) = 0$$

15. **Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 if $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$.**

$$\frac{\partial(y_{1,}y_{2,}y_{3,})}{\partial(x_{1,}x_{2,}x_{3,})} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2\,x_2^2\,x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_2x_1 \\ x_3x_2 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_1x_3 & x_1x_2 \end{vmatrix}$$

$$= \frac{x_1^2\,x_2^2\,x_3^2}{x_1^2\,x_2^2\,x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= (-1)[1-1]-1[-1-1]+1[1+1]$$

$$= 0+2+2=4$$

16. **If** $u = 2xy,\ v = x^2 - y^2,\ x = r\cos\theta,\ y = r\sin\theta$**, prove that** $\frac{\partial(u,v)}{\partial(r,\theta)}$**.**

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)}\,\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$u = 2xy, \qquad v = x^2 - y^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta$$

$$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial v}{\partial x} = 2x \qquad \frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial u}{\partial y} = 2x \qquad \frac{\partial v}{\partial y} = -2y \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= \left(-4y^2 - 4x^2\right)\left(r\cos^2\theta + r\sin^2\theta\right)$$

$$= -4\left(x^2 + y^2\right)r, \qquad \because \cos^2\theta + \sin^2\theta = 1$$

$$= -4r^3, \qquad \because x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

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17. Find the Jacobian of u, v with respect to x, y, If $u = 2xy$ and $v = x^2 - y^2$.

$$u = 2xy \qquad \frac{\partial u}{\partial x} = 2y \qquad \frac{\partial u}{\partial y} = 2x$$

$$v = x^2 - y^2 \quad \frac{\partial v}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = -2y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}$$

$$= -4y^2 - 4x^2 = -4\left(y^2 + x^2\right)$$

18. Are the functions $u = \frac{x+y}{1-xy}$ *and* $v = \tan^{-1} x + \tan^{-1} y$ functionally dependent? If so, Find the relation between them.

$$Given: \ u = \frac{x+y}{1-xy} \qquad\qquad v = \tan^{-1} x + \tan^{-1} y$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)1-(x+y)(-y)}{(1-xy)^2} \quad = \frac{1-xy+xy+y^2}{(1-xy)^2} \quad = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)1-(x+y)(-x)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2} \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1+y^2}{(1-xy)^2}\frac{1}{1+y^2} - \frac{1}{1+x^2}\frac{1+x^2}{(1-xy)^2}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Since $\frac{\partial(u,v)}{\partial(x,y)} = 0$, u *and* v are functionally dependent. Also $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Therefore $v = \tan^{-1} u$ and hence $u = \tan v$

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Exercise

1 If $u = x^2 \ \ and \ \ v = y^2$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$.

2 If $u = xyz, \ v = xy + yz + xz, \ w = x + y + z$, find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

3 Transform the three dimensional Cartesian coordinates (x,y,z) to spherical polar coordinates (r,θ,ϕ) and hence find the Jacobian of x, y, z with respect to r,θ,ϕ.

4 Prove that $u = \sin^{-1} x + \sin^{-1} y; \ v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ are functionally dependent and hence find the relationship between them.

Taylor's series expansion of a function of two variables.

Taylor's series of $f(x,y)$ at or near point (a,b) is

$$f(x,y) = f(a,b) + \frac{1}{1!}\Big[(x-a)f_x(a,b) + (y-b)f_y(a,b)\Big] +$$
$$\frac{1}{2!}\Big[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big] +$$
$$\frac{1}{3!}\Big[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\Big] + \ldots..$$

This is Taylor's expansion of $f(x,y)$ in powers of $(x-a) \ and \ (y-b)$.

Note: Taylor's series of $f(x,y)$ at or near point $(0,0)$ is Maclaurin's series

$$f(x,y) = f(0,0) + \frac{1}{1!}\Big[(x)f_x(0,0) + (y)f_y(0,0)\Big] +$$
$$\frac{1}{2!}\Big[(x)^2 f_{xx}(0,0) + 2(x)(y)f_{xy}(0,0) + (y)^2 f_{yy}(0,0)\Big] +$$
$$\frac{1}{3!}\Big[(x)^3 f_{xxx}(0,0) + 3(x)^2(y)f_{xxy}(0,0) + 3(x)(y)^2 f_{xyy}(0,0) + (y)^3 f_{yyy}(0,0)\Big] + \ldots..$$

1. Find the Taylor series expansion of x^y near the point (1, 1) up to the first degree terms.

Function	Value at (1,1)
$f(x,y) = x^y$	$f(1,1) = 1^1 = 1$
$f_x(x,y) = y.x^{y-1}$	$f_x(1,1) = 1.1^{1-1} = 1$
$f_y(x,y) = x^y.\log y$	$f_y(1,1) = 1^1.\log 1 = 0$

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By Taylors Expansion

$$f(x,y)=f(1,1)+\frac{1}{1!}\left[(x-1)f_x(1,1)+(y-1)f_y(1,1)\right]+.....$$

$$=1+\frac{1}{1!}\left[(x-1)(1)+(y-1)(0)\right]+.....$$

$$=1+(x-1)+.....$$

2. **Expand $e^x \sin y$ in powers of x *and* y as far as terms of the second degree using Taylor's theorem.**

Function	Value at (0,0)
$f(x,y)=e^x \sin y$	$f(0,0)=0$
$f_x=e^x \sin y$	$f_x(0,0)=0$
$f_y=e^x \cos y$	$f_y(0,0)=1$
$f_{xx}=e^x \sin y$	$f_{xx}(0,0)=0$
$f_{xy}=e^x \cos y$	$f_{xy}(0,0)=1$
$f_{yy}=-e^x \sin y$	$f_{yy}(0,0)=0$

Taylor's series of $f(x,y)$ at or near point $(0,0)$ is Maclaurins series

$$f(x,y)=f(0,0)+\frac{1}{1!}\left[(x)f_x(0,0)+(y)f_y(0,0)\right]+$$

$$\frac{1}{2!}\left[(x)^2 f_{xx}(0,0)+2(x)(y)f_{xy}(0,0)+(y)^2 f_{yy}(0,0)\right]+$$

$$\frac{1}{3!}\left[(x)^3 f_{xxx}(0,0)+3(x)^2(y)f_{xxy}(0,0)+3(x)(y)^2 f_{xyy}(0,0)+(y)^3 f_{yyy}(0,0)\right]+.....$$

Substitute all the above values

$$f(x,y)=0+\left[0+y\right]+\frac{1}{2!}\left[x^2(0)+2xy+y^2(0)\right]$$

$$f(x,y)=y+\left[xy\right]=y(1+x)$$

3. **Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's expansion up to the third degree terms.**

Let $f(x,y)=x^2y+3y-2$. Here $a=1, b=-2$.

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Function	*Value at* $(1,-2)$
$f(x,y)=x^2y+3y-2$	$f(1,-2)=-2-6-2=-10$
$f_x(x,y)=2xy$	$f_x(1,-2)=2(1)(-2)=-4$
$f_{xx}(x,y)=2y$	$f_{xx}(1,-2)=2(-2)=-4$
$f_{xxx}(x,y)=0$	$f_{xxx}(1,-2)=0$
$f_y(x,y)=x^2+3$	$f_y(1,-2)=1+3=4$
$f_{yy}(x,y)=0$	$f_{yy}(1,-2)=0$
$f_{yyy}(x,y)=0$	$f_{yyy}(1,-2)=0$
$f_{xy}(x,y)=2x$	$f_{xy}(1,-2)=2$
$f_{xyy}(x,y)=0$	$f_{xyy}(1,-2)=0$
$f_{xxy}(x,y)=2$	$f_{xxy}(1,-2)=2$

Taylor's series of $f(x,y)$ in terms of $(x-1)$ & $(y+2)$ is

$$f(x,y)=f(1,-2)+\frac{1}{1!}\Big[(x-1)f_x(1,-2)+(y+2)f_y(1,-2)\Big]+$$

$$\frac{1}{2!}\Big[(x-1)^2f_{xx}(1,-2)+2(x-1)(y+2)f_{xy}(1,-2)+(y+2)^2f_{yy}(1,-2)\Big]+$$

$$\frac{1}{3!}\Big[(x-1)^3f_{xxx}(1,-2)+3(x-1)^2(y+2)f_{xxy}(1,-2)+3(x-1)(y+2)^2f_{xyy}(1,-2)+(y+2)^3f_{yyy}(1,-2)\Big]+.....$$

$$f(x,y)=-10+\frac{1}{1!}\big[(x-1)(-4)+(y+2)(-4)\big]+$$

$$\frac{1}{2!}\Big[(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2(0)\Big]+$$

$$\frac{1}{3!}\Big[(x-1)^3(0)+3(x-1)^2(y+2)(2)+3(x-1)(y+2)^2(0)+(y+2)^3(0)\Big]+.....$$

4. Expand $e^x\cos y$ about $\left(0,\frac{\pi}{2}\right)$ upto the third term using Taylor's series.

Function	**Value at** $\left(0,\frac{\pi}{2}\right)$
$f(x,y)=e^x\cos y$	$f=0$
$f_x=e^x\cos y$ $f_y=-e^x\sin y$	$f_x=0$ $f_y=-1$
$f_{xx}=e^x\cos y$ $f_{xy}=-e^x\sin y$ $f_{yy}=-e^x\cos y$	$f_{xx}=0$ $f_{xy}=-1$ $f_{yy}=0$
$f_{xxx}=e^x\cos y$ $f_{xxy}=-e^x\sin y$ $f_{xyy}=-e^x\cos y$	$f_{xxx}=0$ $f_{xxy}=-1$ $f_{xyy}=0$

$f_{yyy}=e^{x}\sin y$	$f_{yyy}=1$

Taylor's series of $f(x,y)$ at $\left(0,\frac{\pi}{2}\right)$ is

$$f(x,y)=f\left(0,\frac{\pi}{2}\right)+\frac{1}{1!}\left[(x)f_x\left(0,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)f_y\left(0,\frac{\pi}{2}\right)\right]+$$

$$\frac{1}{2!}\left[(x)^2 f_{xx}\left(0,\frac{\pi}{2}\right)+2(x)\left(y-\frac{\pi}{2}\right)f_{xy}\left(0,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^2 f_{yy}\left(0,\frac{\pi}{2}\right)\right]+$$

$$\frac{1}{3!}\left[(x)^3 f_{xxx}\left(0,\frac{\pi}{2}\right)+3(x)^2\left(y-\frac{\pi}{2}\right)f_{xxy}\left(0,\frac{\pi}{2}\right)+3(x)\left(y-\frac{\pi}{2}\right)^2 f_{xyy}\left(0,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^3 f_{yyy}\left(0,\frac{\pi}{2}\right)\right]+.....$$

$$f(x,y)=0+\frac{1}{1!}\left[(x)(0)+\left(y-\frac{\pi}{2}\right)(-1)\right]+\frac{1}{2!}\left[x^2(0)+2x\left(y-\frac{\pi}{2}\right)(-1)+\left(y-\frac{\pi}{2}\right)^2(0)\right]$$

$$+\frac{1}{3!}\left[x^3(0)+3x^2\left(y-\frac{\pi}{2}\right)(-1)+3x\left(y-\frac{\pi}{2}\right)^2(0)+\left(y-\frac{\pi}{2}\right)^3(1)\right]+\ldots$$

$$=-y+\frac{\pi}{2}+\frac{1}{2!}\left[-2xy+2x\frac{\pi}{2}\right]+\frac{1}{3!}\left[-3x^2y+\frac{3\pi}{2}x^2+\left(y-\frac{\pi}{2}\right)^3\right]+\ldots$$

5. Expand $e^{x}\log y$ as Taylor's series in powers of x and $(y-1)$ upto third degree terms.

Function	**Value at (0,1)**
$f(x,y)=e^{x}\log y$	$f(0,1)=e^{0}\log 1=0$
$f_x(x,y)=e^{x}\log y$ $f_y(x,y)=e^{x}.\frac{1}{y}$	$f_x(0,1)=e^{0}\log 1=0$ $f_y(0,1)=e^{0}.\frac{1}{1}=1$
$f_{xx}(x,y)=e^{x}\log y$ $f_{xy}(x,y)=e^{x}.\frac{1}{y}$ $f_{yy}(x,y)=-e^{x}.\frac{1}{y^2}$	$f_{xx}(0,1)=e^{0}\log 1=0$ $f_{xy}(0,1)=e^{0}.\frac{1}{1}=1$ $f_{yy}(0,1)=-e^{0}.\frac{1}{1}=-1$

$f_{xxx}(x,y)=e^x \log y$ $f_{xxy}(x,y)=e^x.\frac{1}{y}$ $f_{xyy}(x,y)=-e^x.\frac{1}{y^2}$ $f_{yyy}(x,y)=2e^x.\frac{1}{y^3}$	$f_{xxx}(0,1)=e^0 \log 1=0$ $f_{xxy}(0,1)=e^0.\frac{1}{1}=1$ $f_{xyy}(0,1)=-e^0.\frac{1}{1}=-1$ $f_{yyy}(0,1)=2e^0.\frac{1}{1}=2$

$$\begin{aligned} f(x,y) = f(0,1) &+ \frac{1}{1!}\left[(x)f_x(0,1)+(y-1)f_y(0,1)\right]+ \\ &\frac{1}{2!}\left[(x)^2 f_{xx}(0,1)+2(x)(y-1)f_{xy}(0,1)+(y-1)^2 f_{yy}(0,1)\right]+ \\ &\frac{1}{3!}\left[(x)^3 f_{xxx}(0,1)+3(x)^2(y-1)f_{xxy}(0,1)+3(x)(y-1)^2 f_{xyy}(0,1)+(y-1)^3 f_{yyy}(0,1)\right]+..... \end{aligned}$$

$$\begin{aligned} \mathrm{e}^{\mathrm{x}} \log \mathrm{y} &= 0+\frac{1}{1!}\left[\mathrm{x}(0)+(\mathrm{y}-1)(1)\right]+\frac{1}{2!}\left[\mathrm{x}^2(0)+2\mathrm{x}(\mathrm{y}-1)(1)+(\mathrm{y}-1)^2(-1)\right] \\ &+\frac{1}{3!}\left[\mathrm{x}^3(0)+3\mathrm{x}^2(\mathrm{y}-1)(1)+3\mathrm{x}(\mathrm{y}-1)^2(-1)+(\mathrm{y}-1)^3(2)\right]+.... \\ &=(\mathrm{y}-1)+\frac{1}{2!}\left[+2\mathrm{x}(\mathrm{y}-1)-(\mathrm{y}-1)^2\right]+\frac{1}{3!}\left[3\mathrm{x}^2(\mathrm{y}-1)-3\mathrm{x}(\mathrm{y}-1)^2+2(\mathrm{y}-1)^3\right]+.... \end{aligned}$$

6. Expand $e^x \log(1+y)$ in powers of x and y upto terms of third degree using Taylor's theorem.

Function	**Value at (0, 0)**
$f(x,y)=e^x \log(1+y)$	$f=0$
$f_x=e^x \log(1+y)$ $f_y=e^x(1+y)^{-1}$	$f_x=0$ $f_y=1$
$f_{xx}=e^x \log(1+y)$ $f_{xy}=e^x(1+y)^{-1}$ $f_{yy}=-e^x(1+y)^{-2}$	$f_{xx}=0$ $f_{xy}=1$ $f_{yy}=-1$
$f_{xxx}=e^x \log(1+y)$ $f_{xxy}=e^x(1+y)^{-1}$ $f_{xyy}=-e^x(1+y)^{-2}$ $f_{yyy}=2e^x(1+y)^{-3}$	$f_{xxx}=0$ $f_{xxy}=1$ $f_{xyy}=-1$ $f_{yyy}=2$

Taylor's series of $f(x,y)$ at $(0,0)$ is

$$f(x,y)=f(0,0)+\frac{1}{1!}\Big[(x)f_x(0,0)+(y)f_y(0,0)\Big]+$$

$$\frac{1}{2!}\Big[(x)^2 f_{xx}(0,0)+2(x)(y)f_{xy}(0,0)+(y)^2 f_{yy}(0,0)\Big]+$$

$$\frac{1}{3!}\Big[(x)^3 f_{xxx}(0,0)+3(x)^2(y)f_{xxy}(0,0)+3(x)(y)^2 f_{xyy}(0,0)+(y)^3 f_{yyy}(0,0)\Big]+.....$$

$$e^x\log(1+y)=0+\frac{x(0)+y(1)}{1!}+\frac{x^2(0)+2xy(1)+y^2(-1)}{2!}$$

$$+\frac{x^3(0)+3x^2y(1)+3xy^2(-1)+y^3(2)}{3!}+....$$

$$=\frac{y}{1!}+\frac{2xy-y^2}{2!}+\frac{3x^2y-3xy^2+2y^3}{3!}+....$$

$$=y+\frac{2xy-y^2}{2}+\frac{3x^2y-3xy^2+2y^3}{6}+....$$

7. Expand the Taylor's series function $f(x,y)=x^2y^2+2x^2y+3xy^2$ of $(x+2)$ and $(y-1)$ up to the third powers.

Function	*Value at* $(-2,1)$
$f(x,y)=x^2y^2+2x^2y+3xy^2$	$f(-2,1)=2+8-6=4$
$f_x(x,y)=2xy^2+4xy+3y^2$	$f_x(-2,1)=-4-8+3=-9$
$f_{xx}(x,y)=2y^2+4y$	$f_{xx}(-2,1)=2+4=6$
$f_{xxx}(x,y)=0$	$f_{xxx}(-2,1)=0$
$f_y(x,y)=2x^2y+2x^2+6xy$	$f_y(-2,1)=8+8-12=4$
$f_{yy}(x,y)=2x^2+6x$	$f_{yy}(-2,1)=8-12=-4$
$f_{yyy}(x,y)=0$	$f_{yyy}(-2,1)=0$
$f_{xy}(x,y)=4xy+4x+6y$	$f_{xy}(-2,1)=-8-8+6=-10$
$f_{xyy}(x,y)=4x+6$	$f_{xyy}(-2,1)=-8+6=-2$
$f_{xxy}(x,y)=4y+4$	$f_{xxy}(-2,1)=4+4=8$

Taylor's series of $f(x,y)$ in terms of $(x+2)$ & $(y-1)$ is

$$f(x,y)=f(-2,1)+\frac{1}{1!}\Big[(x+2)f_x(-2,1)+(y-1)f_y(-2,1)\Big]+$$

$$\frac{1}{2!}\Big[(x+2)^2 f_{xx}(-2,1)+2(x+2)(y-1)f_{xy}(-2,1)+(y-1)^2 f_{yy}(-2,1)\Big]+$$

$$\frac{1}{3!}\Big[(x+2)^3 f_{xxx}(-2,1)+3(x+2)^2(y-1)f_{xxy}(-2,1)+3(x+2)(y-1)^2 f_{xyy}(-2,1)+(y-1)^3 f_{yyy}(-2,1)\Big]+.....$$

$$x^2y^2+2x^2y+3xy^2=6+\frac{1}{1!}[(x+2)(-9)+(y-1)(4)]+\frac{1}{2!}[(x+2)^2(6)+2(x+2)(y-1)(-10)+(y-1)^2(-4)]$$

$$+\frac{1}{3!}[(x+2)^3(0)+3(x+2)^2(y-1)(8)+3(x+2)(y-1)^2(-2)+(y-1)^3(0)]+....$$

8. Expand $\sin xy$ at $\left(1,\frac{\pi}{2}\right)$ up to second degree terms using Taylor's series.

Given function is $f\left(x,y\right)=\sin xy$

Function	Value at $\left(1,\frac{\pi}{2}\right)$
$f\left(x,y\right)=\sin xy$	$f=\sin\frac{\pi}{2}=1$
$f_x=y\cos xy$ $f_y=x\cos xy$	$f_x=\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2}=0$ $f_y=\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2}=0$
$f_{xx}=-y^2\sin xy$ $f_{xy}=-xy\sin xy+\cos xy$ $f_{yy}=-x^2\sin xy$	$f_{xx}=-\left(\frac{\pi}{2}\right)^2\sin\frac{\pi}{2}=\frac{\pi^2}{4}$ $f_{xy}=-\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2}+\cos\frac{\pi}{2}=-\frac{\pi}{2}$ $f_{yy}=-\left(1\right)^2\sin\frac{\pi}{2}=-1$

Taylor's series of $f(x,y)$ at $\left(1,\frac{\pi}{2}\right)$ is

$$f(x,y)=f\left(1,\frac{\pi}{2}\right)+\frac{1}{1!}\left[(x-1)f_x\left(1,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)f_y\left(1,\frac{\pi}{2}\right)\right]+$$

$$\frac{1}{2!}\left[(x-1)^2f_{xx}\left(1,\frac{\pi}{2}\right)+2(x-1)\left(y-\frac{\pi}{2}\right)f_{xy}\left(1,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^2f_{yy}\left(1,\frac{\pi}{2}\right)\right]+\ldots$$

$$f\left(x,y\right)=1+0+0+\frac{1}{2!}\left[\left(x-1\right)^2\left(-\frac{\pi^2}{4}\right)+2\left(x-1\right)\left(y-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^2\left(-1\right)\right]+\ldots$$

$$=1+-\frac{\pi^2}{8}\left(x-1\right)^2-\frac{\pi}{2}\left(x-1\right)\left(y-\frac{\pi}{2}\right)-\left(y-\frac{\pi}{2}\right)^2+\ldots$$

9. **Find the Taylor's series expansion of $e^x \cos y$ at $\left(1, \frac{\pi}{4}\right)$ upto the third degree terms.**

Function	Value at $\left(1, \frac{\pi}{4}\right)$
$f(x,y) = e^x \cos y$	$f = \frac{e}{\sqrt{2}}$
$f_x = e^x \cos y$ $f_y = -e^x \sin y$	$f_x = \frac{e}{\sqrt{2}}$ $f_y = -\frac{e}{\sqrt{2}}$
$f_{xx} = e^x \cos y$ $f_{xy} = -e^x \sin y$ $f_{yy} = -e^x \cos y$	$f_{xx} = \frac{e}{\sqrt{2}}$ $f_{xy} = -\frac{e}{\sqrt{2}}$ $f_{yy} = -\frac{e}{\sqrt{2}}$
$f_{xxx} = e^x \cos y$ $f_{xxy} = -e^x \sin y$ $f_{xyy} = -e^x \cos y$ $f_{yyy} = e^x \sin y$	$f_{xxx} = \frac{e}{\sqrt{2}}$ $f_{xxy} = -\frac{e}{\sqrt{2}}$ $f_{xyy} = -\frac{e}{\sqrt{2}}$ $f_{yyy} = \frac{e}{\sqrt{2}}$

Taylor's series of $f(x,y)$ at $\left(1, \frac{\pi}{4}\right)$ is

$$f(x,y) = f\left(1,\frac{\pi}{4}\right) + \frac{1}{1!}\left[(x-1) f_x\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right) f_y\left(1,\frac{\pi}{4}\right)\right] +$$

$$\frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1,\frac{\pi}{4}\right) + 2(x-1)\left(y-\frac{\pi}{4}\right) f_{xy}\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)^2 f_{yy}\left(1,\frac{\pi}{4}\right)\right] +$$

$$\frac{1}{3!}\left[(x-1)^3 f_{xxx}\left(1,\frac{\pi}{4}\right) + 3(x-1)^2\left(y-\frac{\pi}{4}\right) f_{xxy}\left(1,\frac{\pi}{4}\right) + 3(x-1)\left(y-\frac{\pi}{4}\right)^2 f_{xyy}\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)^3 f_{yyy}\left(1,\frac{\pi}{4}\right)\right] + \ldots..$$

$$f(x,y) = \frac{e}{\sqrt{2}} + \frac{1}{1!}\left[(x-1)\frac{e}{\sqrt{2}} - \left(y-\frac{\pi}{4}\right)_y \frac{e}{\sqrt{2}}\right] +$$

$$\frac{1}{2!}\left[(x-1)^2 \frac{e}{\sqrt{2}} - 2(x-1)\left(y-\frac{\pi}{4}\right)\frac{e}{\sqrt{2}} - \left(y-\frac{\pi}{4}\right)^2 \frac{e}{\sqrt{2}}\right] +$$

$$\frac{1}{3!}\left[(x-1)^3 \frac{e}{\sqrt{2}} - 3(x-1)^2\left(y-\frac{\pi}{4}\right)\frac{e}{\sqrt{2}} - 3(x-1)\left(y-\frac{\pi}{4}\right)^2 \frac{e}{\sqrt{2}} + \left(y-\frac{\pi}{4}\right)^3 \frac{e}{\sqrt{2}}\right] + \ldots..$$

10. Obtain the Taylor's series of $x^3 + y^3 + xy^2$ at $(1,2)$.

Function	**Value at (1, 2)**
$f(x,y) = x^3 + y^3 + xy^2$	$f(1,2) = 1^3 + 2^3 + (1)(2)^2 = 1 + 8 + 4 = 13$
$f_x = 3x^2 + y^2$ $f_y = 3y^2 + 2xy$	$f_x = 3(1)^2 + 2^2 = 3 + 4 = 7$ $f_y = 3(2)^2 + 2(1)(2) = 12 + 4 = 16$
$f_{xx} = 6x$ $f_{xy} = 2y$ $f_{yy} = 6y + 2x$	$f_{xx} = 6(1) = 6$ $f_{xy} = 2(2) = 4$ $f_{yy} = 6(2) + 2(1) = 12 + 2 = 14$
$f_{xxx} = 6$ $f_{xxy} = 0$ $f_{xyy} = 2$ $f_{yyy} = 6$	$f_{xxx} = 6$ $f_{xxy} = 0$ $f_{xyy} = 2$ $f_{yyy} = 6$

Taylor's series of $f(x,y)$ in terms of $(x-1)$ & $(y-2)$ is

$$f(x,y) = f(1,2) + \frac{1}{1!}\left[(x-1)f_x(1,2) + (y-2)f_y(1,2)\right] +$$
$$\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,2) + 2(x-1)(y-2)f_{xy}(1,2) + (y-2)^2 f_{yy}(1,2)\right] +$$
$$\frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,2) + 3(x-1)^2(y-2)f_{xxy}(1,2) + 3(x-1)(y-2)^2 f_{xyy}(1,2) + (y-2)^3 f_{yyy}(1,2)\right] + ...$$

$$\therefore\ f(x,y) = 13 + \frac{1}{1!}\left[(x-1)7 + (y-2)16\right] + \frac{1}{2!}\left[(x-1)^2 6 + 2(x-1)(y-2)4 + (y-2)^2 14\right]$$
$$+ \frac{1}{3!}\left[(x-1)^3 6 + 3(x-1)^2(y-2)(0) + 3(x-1)(y-2)^2 2 + (y-2)^3 6\right] + ...$$

$$= 13 + 7(x-1) + 16(y-2) + \frac{1}{2}\left[6(x-1)^2 + 8(x-1)(y-2) + 14(y-2)^2\right]$$
$$+ \frac{1}{6}\left[6(x-1)^3 + 6(x-1)(y-2)^2 + 6(y-2)^3\right] + ...$$

$$\therefore f(x,y) = 13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 4(x-1)(y-2) + 7(y-2)^2$$
$$+ (x-1)^3 + (x-1)(y-2)^2 + (y-2)^3$$

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11. Use Taylor formula to expand the function f defined by $f(x,y)=x^2+xy+y^2$ in powers of $(x-1)$ and $(y-2)$

Function	*Value at* $(1,2)$
$f(x,y)=x^2+xy+y^2$	$f(1,2)=1+2+4$
$f_x(x,y)=2x+y$	$f_x(1,2)=2+2=4$
$f_y(x,y)=x+2y$	$f_y(1,2)=1+4=5$
$f_{xx}(x,y)=x$	$f_{xx}(1,2)=1$
$f_{xy}(x,y)=1$	$f_{xy}(1,2)=1$
$f_{yy}(x,y)=2$	$f_{yy}(1,2)=2$

Taylor's series of $f(x,y)$ in terms of $(x-1)$ & $(y-2)$ is

$$f(x,y)=f(1,2)+\frac{1}{1!}\left[(x-1)f_x(1,2)+(y-2)f_y(1,2)\right]+$$

$$\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,2)+2(x-1)(y-2)f_{xy}(1,2)+(y-2)^2 f_{yy}(1,2)\right]+....$$

$$f(x,y)=4+\frac{1}{1!}\left[(x-1)4+(y-2)5\right]+\frac{1}{2!}\left[(x-1)^2+2(x-1)(y-2)+(y-2)^2 2\right]+....$$

Exercise

1 Find the Taylor's series expansion of e^{xy} at (1,1) up to the third degree terms.

2 Find the Taylor's series expansion of y^x at (1,1) up to the third degree terms.

3 Expand $x^2y+3y-2$ in powers of $(x-1)$ *and* $(y+2)$ up to third degree terms.

4 Expand $x^2y+2x-3y$ in powers of $(x+2)$ *and* $(y-1)$ up to third degree terms.

5 Expand $\tan^{-1}\frac{y}{x}$ in powers of $(x-1)$ *and* $(y-1)$ up to third degree terms.

Homogeneous Functions

An expression in which every term is of n^{th} degree is called a homogeneous function of degree n. This can be expressed as $x^n f\left(\frac{y}{x}\right)$. This can be extended for any number of variables.

Euler's Theorem on homogeneous functions

1. If u is a homogeneous function of degree n in x & y then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

Note: If u is a homogeneous function of degree n in x, y & z then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$

2. If u is a homogeneous function of degree n in x & y then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

Solved Problems on Euler's Theorem

1. **If** $u=\log\left(x^2+xy+y^2\right)$ **prove that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2.$

Given that $u=\log\left(x^2+xy+y^2\right)$

Here u is not a homogeneous function

Taking log on both sides in equation (1) we get

$$\therefore e^u=\left(x^2+xy+y^2\right)$$

To check homogenous:

$$\therefore e^u=\left(t^2x^2+t^2xy+t^2y^2\right)=t^2\left(x^2+xy+y^2\right)$$

e^u is a homogeneous function of degree 2 in x & y

$\therefore$ By Euler's theorem $\mathrm{x}\frac{\partial \mathrm{f}}{\partial \mathrm{x}}+\mathrm{y}\frac{\partial \mathrm{f}}{\partial \mathrm{x}}=\mathrm{nf}$, we have

$$x\frac{\partial}{\partial x}(e^u)+y\frac{\partial}{\partial y}(e^u)=2(e^u).$$

$$xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=2e^u.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2.$$

2. **If** $u=\tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ **show that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u$ **.**

Given $u=\tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$$\tan u=\frac{x^2+y^2}{x+y} \qquad -----(1)$$

Let $f(x,y)=\tan u=\frac{x^2+y^2}{x+y}$

$$f(tx,ty)=\frac{(tx)^2+(ty)^2}{tx+ty}=t\left[\frac{x^2+y^2}{x+y}\right]=tf(x,y)$$

$\therefore$ $f(x,y)$ is a homogeneous function in x and y of degree 1.

By Euler's theorem $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$

$$\Rightarrow x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=\tan u$$

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$$\Rightarrow \sec^2 u\, x\frac{\partial u}{\partial x} + \sec^2 u\, y\frac{\partial u}{\partial y} = \tan u$$

$$\Rightarrow \sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u}$$

$$= \frac{\sin u}{\cos u}\cos^2 u \qquad \left[\because \tan u = \frac{\sin u}{\cos u}, \sec u = \frac{1}{\cos u}\right]$$

$$= \sin u \cos u$$

$$= \frac{1}{2}\sin 2u \qquad [\because \sin 2u = 2\sin u \cos u]$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$$

3. **If** $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$**.**

Given that $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$f = \tan u = \left(\frac{x^3+y^3}{x-y}\right)$$

$$f(tx, ty) = \left(\frac{t^3x^3+t^3y^3}{tx-ty}\right) = \left(\frac{t^3}{t}\frac{x^3+y^3}{x-y}\right) = \left(t^2\frac{x^3+y^3}{x-y}\right)$$

Here $f = \tan u$ is a homogeneous function of degree 2 in x and y

By Euler's theorem $\mathrm{x}\frac{\partial \mathrm{f}}{\partial \mathrm{x}} + \mathrm{y}\frac{\partial \mathrm{f}}{\partial \mathrm{x}} = \mathrm{nf}$

$$x\frac{\partial}{\partial x}\tan u + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u\frac{1}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}.\cos^2 u$$

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$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\sin u\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u \qquad \because \sin 2u=2\sin u\cos u$$

4. **If** $\mathbf{u=\tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)}$**. show that** $\mathbf{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{4}\sin 2u}$

Take $\tan u=\frac{x+y}{\sqrt{x}+\sqrt{y}}=f(x,y)$

$$f(xt,yt)=\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}=t^{\frac{1}{2}}\frac{x+y}{\sqrt{x}+\sqrt{y}}=t^{\frac{1}{2}}f(x,y)$$

f(x, y) is homogeneous function of degree $\frac{1}{2}$

By Eulers theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

$$x\frac{\partial(\tan u)}{\partial x}+y\frac{\partial(\tan u)}{\partial y}=\frac{1}{2}\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u\frac{1}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\frac{\sin u}{\cos u}\cos^2 u=\frac{1}{2}\sin u\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin u\cos u$$

5. **If u =** $\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ **, show that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$

Given: $u=\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

$\sin u=\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}=f(x,y)$ say

$$f(xt,yt)=\frac{\sqrt{xt}-\sqrt{yt}}{\sqrt{xt}+\sqrt{yt}}=\frac{\sqrt{t}}{\sqrt{t}}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}=t^0\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

$f(x,y)$ is a homogeneous of degree n=0

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By Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial x}=nf$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial x}=0$$

$$x\frac{\partial (\sin u)}{\partial x}+y\frac{\partial (\sin u)}{\partial y}=0$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=0$$

divide by cos u on both sides

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Hence proved

6. **If** $u=\log\left[\frac{x^2+y^2}{x+y}\right]$**, prove that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$

Given $u=\log\left[\frac{x^2+y^2}{x+y}\right]$

$$f=e^u=\left[\frac{x^2+y^2}{x+y}\right]$$

Here f is homogeneous of degree 1. By Euler theorem,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f \quad where \quad f=e^u$$

$$xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=e^u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$$

7. **If** $u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$**, prove that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$

Given $u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$$f=\sin u=\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

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Here f is homogeneous of degree 1/2. By Euler theorem,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{1}{2}f \quad where \quad f=\sin u$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\ \tan u$$

8. **If** $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ **prove that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$ **.**

$$Given\ \ u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

Let $f(x,\ y)=\sin u\ =\frac{x^2+y^2}{x+y}$

$f(tx,\ ty)=\frac{(tx)^2+(ty)^2}{tx+ty}$ =t$\frac{x^2+y^2}{x+y}$ =tf(x,y)

$\therefore f(x,y)=\sin u$ is homogeneous function in x and y of degree 1.

By Euler's theorem $\mathrm{x}\frac{\partial \mathrm{f}}{\partial \mathrm{x}}+\mathrm{y}\frac{\partial \mathrm{f}}{\partial \mathrm{x}}=\mathrm{nf}$

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=\sin u$$

$$\cos ux\frac{\partial u}{\partial x}+\cos uy\frac{\partial u}{\partial y}=\sin u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$$

9. **If** $u=\cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ **Prove that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$ **.**

Given

$$u=\cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

$$\cos u=\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

Let $f(x,y)=\cos u=\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

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$$f(tx,ty)=\left[\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}\right]=t^{\frac{1}{2}}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]=t^{\frac{1}{2}}f(x,y)$$

$f(x,y)$ is a homogeneous function in x and y of degree $\frac{1}{2}$.

$$By\ Euler's\ theorem\ x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

$$x\frac{\partial(\cos u)}{\partial x}+y\frac{\partial(\cos u)}{\partial y}=n(\cos u)$$

$$-\sin u.x\frac{\partial u}{\partial x}-\sin u.\mathrm{y}\frac{\partial u}{\partial y}=\frac{1}{2}\cos u$$

$$-\sin u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=\frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\frac{1}{2}\cos u}{-\sin u}\quad\left[\therefore\frac{\cos u}{\sin u}=\cot u\right]$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$$

10. If $u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$, **prove that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$ **and** $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}$.

Given $u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$$f(x,y)=\sin u=\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

$$f(tx,ty)=\left[\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}\right]=\frac{t}{\sqrt{t}}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]=t^{\frac{1}{2}}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

Therefore $f(x,y)$ is a homogeneous function of degree $\frac{1}{2}$.

Therefore by Euler's theorem $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{1}{2}f\quad where\ \ f=\sin u$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u$$

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Divide by $\cos u$, we have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$ .....(1)

Differentiate (1) partially w.r.t x

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u.\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}....(2)$$

Differentiate (1) partially w.r.t y

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u.\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial y}....(3)$$

Multiply (2) by x and (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=x\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}+y\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial y}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(\frac{1}{2}\tan u\right)$$

$$=\left(\frac{1}{2}\frac{1}{\cos^2 u}-1\right)\left(\frac{1}{2}\frac{\sin u}{\cos u}\right)$$

$$=\left(\frac{1}{4}\frac{\sin u}{\cos^3 u}-\frac{1}{2}\frac{\sin u}{\cos u}\right)$$

$$=-\frac{\sin u}{4\cos^3 u}\left[-1+2\cos^2 u\right]$$

$$=-\frac{\sin u\cos 2u}{4\cos^3 u},\quad \because \cos^2 u=\frac{1+\cos 2u}{2}$$

11. **If** $u=\tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$, **prove that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$ **and** $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\cos 3u\sin u$.

Given $u=\tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$

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$$f(x,y)=\tan u=\left[\frac{x^3+y^3}{x-y}\right]$$

$$f(tx,ty)=\left[\frac{t^3x^3+t^3y^3}{tx-ty}\right]=\frac{t^3}{t}\left[\frac{x^3+y^3}{x-y}\right]=t^2\left[\frac{x^3+y^3}{x-y}\right]$$

Therefore $f(x,y)$ is a homogeneous function of degree 2.

Therefore by Euler's theorem $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=2f \quad where \;\; f=ta\text{n}u$

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2ta\text{n}u$$

Divide by $\sec^2 u$, we have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{\tan u}{\sec^2 u}=2\sin u\cos u=\sin 2u$ .....(1)

Differentiate (1) partially w.r.t x

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u.\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left(2\cos 2u-1\right)\frac{\partial u}{\partial x}....(2)$$

Differentiate (1) partially w.r.t y

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=2\cos 2u.\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=\left(2\cos 2u-1\right)\frac{\partial u}{\partial y}....(3)$$

Multiply (2) by x and (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=x\left(2\cos 2u-1\right)\frac{\partial u}{\partial x}+y\left(2\cos 2u-1\right)\frac{\partial u}{\partial y}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(2\cos 2u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(2\cos 2u-1\right)\left(\sin 2u\right)$$

$$=\left(2\cos 2u-1\right)\left(\sin 2u\right)$$

$$=\left(2\cos 2u\sin 2u-\sin 2u\right)$$

$$=\sin 4u-\sin 2u, \quad \because \sin 2x=2\sin x\cos x$$

$$=2\cos 3u\sin u, \quad \because \sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

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Exercise

1 If $\log u = \dfrac{x^3+y^3}{3x+4y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$

2 If $u = \sin^{-1}\dfrac{x+2y+3z}{x^8+y^8+z^8}$, find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$.

3 Verify Euler theorem for $f(x,y) = 3x^2yz + 5xy^2z + 4z^4$

4 If $\sin u = \dfrac{x^2y^2}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$

5 If $u = \tan^{-1}\left[\dfrac{y^2}{x}\right]$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\sin^2 u\sin 2u$.

Extreme Values

Let f be a function of two variables $x \ and \ y$ and let R be a set contained in the domain of f. Then f has maximum value on R at (x_0, y_0) if $f(x,y) \le f(x_0,y_0)$.
Also f has minimum value on R at (x_0, y_0) if $f(x,y) \ge f(x_0,y_0)$.

Further, f has relative maximum value at (x_0, y_0) if there is a disc D centered at (x_0, y_0) and contained in the domain of f such that $f(x,y) \le f(x_0,y_0)$ for all (x,y) in D.
Also f has relative minimum value at (x_0, y_0) if there is a disc D centered at (x_0, y_0) and contained in the domain of f such that $f(x,y) \ge f(x_0,y_0)$ for all (x,y) in D.

Note: A maximum or minimum value of a function is called its extreme value
Note: Let f have a relative extreme value at (x_0, y_0). If f has partial derivatives at (x_0, y_0), then $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$. We say that f has a critical point at (x_0, y_0) in the domain of f if $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$, or if one of the partial derivatives does not exist.

Hence the **necessary conditions** for $f(x,y)$ to have a maximum or minimum at (x_0, y_0) are $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$.

Stationary Value: $f(x_0,y_0)$ is said to be a stationary value of $f(x,y)$, if $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$. i.e. the function is stationary at that point.

Note: Every extreme value is a stationary value but the converse may not be true.

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Example: Let $f(x,y)=3-x^2+2x-y^2-4y$. Find all critical points of f.

Now $f_x=-2x+2$ and $f_y=-2y-4$

If $f_x=0$ *then* $-2x+2=0$ *i.e.* $x=1$. If $f_y=0$ *then* $-2y-4=0$ *i.e.* $y=-2$

Therefore $(1,-2)$ is the only critical point of f.

Example: Let $f(x,y)=\sqrt{x^2+y^2}$. Find all critical points and all relative extreme values of f.

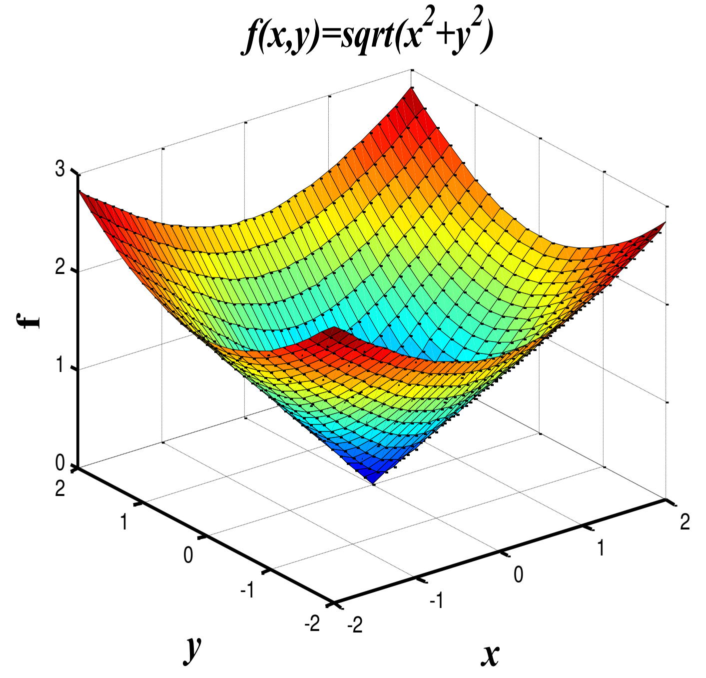
Here

$$f_x(x,y)=\frac{x}{\sqrt{x^2+y^2}} \quad and$$

$$f_y(x,y)=\frac{y}{\sqrt{x^2+y^2}}$$

Since the partial derivatives exists for all points except the origin. Hence $(0,0)$ is a critical point of f.

Also note that no point (x,y) such that $f_x=0$ *and* $f_y=0$.



Since $f(0,0)=0$ *and* $f(x,y)\geq 0$ *for all* (x,y), it follows that 0 is the only relative minimum value of f, and there is no maximum value.

Note: A function need not have a relative extreme value at a critical point.

Example: Let $f(x,y)=y^2-x^2$. Show that the origin is the only critical point but there exists no relative value of f.

Here $f_x(x,y)=-2x$ *and* $f_y(x,y)=2y$

If $f_x=0$ *then* $x=0$. If $f_y=0$ *then* $y=0$

Therefore $(0,0)$ is the only critical point of f.

However $f(0,0)=0$ is not a relative extreme value of f. Because $f(x,0)=-x^2<0$ *for* $x\neq 0$, and $f(0,y)=y^2>0$ *for* $y\neq 0$. Hence f has no relative extreme values.

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Working Rule

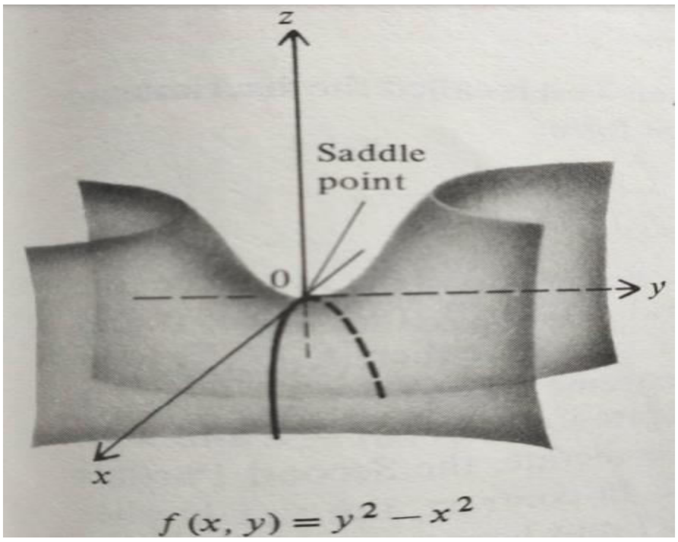
1. Let $f(x,y)$ be a given function. Find $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$.

2. Solve the equations $\frac{\partial f}{\partial x}=0$ *and* $\frac{\partial f}{\partial y}=0$ simultaneously and let the solutions be $(a,b),\ (c,d),\ \ldots\ldots.,$ called stationary points.

3. For each solution (a,b), find the values of $A=\frac{\partial^2 f}{\partial x^2},\ B=\frac{\partial^2 f}{\partial x\partial y},\ C=\frac{\partial^2 f}{\partial y^2}$ *and* $\Delta = AC-B^2.$

4. Consider a solution, say (a,b). then function $f(x,y)$

has maximum at (a,b) if $\Delta>0$ and $A\ or\ C<0$	has minimum at (a,b) if $\Delta>0$ and $A\ or\ C>0$	has neither maximum nor minimum at (a,b) if $\Delta<0$. Then (a,b) is said to be saddle point.	may or may not have relative extreme at (a,b) if $\Delta=0$ i.e. further investigation is required



5. Similarly examine the other pairs of values for extremum of $f(x,y)$.

Solved Problems on Extreme Values

1. Find the minimum point of $f(x,y)=x^2+y^2+6x+12.$

Given $f(x,y)=x^2+y^2+6x+12$

$f_x=2x+6,\quad f_y=2y$

$f_{xx}=2,\quad f_{yy}=2,\quad f_{xy}=0$

The stationary values are given by $f_x=0, f_y=0$

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$f_x = 0 \Rightarrow 2x + 6 = 0 \Rightarrow x = \frac{-6}{2} = -3$

$f_y = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$

$\therefore (-3,0)$ *is the extreme value of* $f(x,y)$

$A = f_{xx} = 2, B = f_{xy} = 0, C = f_{yy} = 2,$

$AC - B^2 = 4 > 0$ and A > 0

(−3,0) is a minimum point.

2. Find the maximum minimum of the values $x^2 - xy + y^2 - 2x + y$

Given function is $f(x,y) = x^2 - xy + y^2 - 2x + y$

$f_x(x,y) = 2x - y - 2$

$f_y(x,y) = -x + 2y + 1$

$f_x = 0$ $f_y = 0$

$2x - y - 2 = 0 \quad ....(1)$ $-x + 2y + 1 = 0 \quad ....(2)$

$(1) + 2\times(2) \Rightarrow 3y = 0$

$y = 0$

substitute y =0 in (1), $2x - 2 = 0$

$x = 1$

(1,0) is the extreme point of $f(x,y)$

$A = f_{xx}(x,y) = 2$

Here $B = f_{xy}(x,y) = -1$

$C = f_{yy}(x,y) = 2$

Now $\Delta = (AC - B^2) = 2.2 - (-1)^2 = 4 - 1 = 3 > 0$ *also* $A > 0$

$\therefore$ (1,0) is a minimum point

$\therefore$ Minimum value is $f(1,0) = 1^2 - 0 + 0 - 2 + 0 = -1$

3. A flat circular plate is heated so that the temperature at any point (x,y) **is** $u = x^2 + 2y^2 - x$**. Find the coldest point on the plates.**

$u = x^2 + 2y^2 - x$

$u_x = 2x - 1$

$u_y = 4y$

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Consider

$u_x = 0, \qquad u_y = 0$

$2x-1=0$, $4y=0$

$x=\frac{1}{2}, \qquad \mathrm{y}=0$

Therefore $\left(\frac{1}{2},0\right)$ is the stationary point.

At the point $\left(\frac{1}{2},0\right)$

$\mathrm{A}=\mathrm{u}_{xx}=2, C=\mathrm{u}_{yy}=4, \ B=\mathrm{u}_{xy}=0$

$\Delta = AC-B^2>0$

u *is* minimum *at* $\left(\frac{1}{2},0\right)$ *and its* minimum *value is* $\frac{-1}{4}$.

4. Find the stationary points of the function $f(x,y)=x^3-y^3-3xy$ **.**

Given: $f(x,y)=x^3-y^3-3xy$

$\mathrm{f_x = 3x^2-3y} \qquad \& \qquad \mathrm{f_y=-3y^2-3x}$

Let $\mathrm{f_x}=0 \qquad \& \qquad$ Let $\mathrm{f_y}=0$

$\mathrm{3x^2-3y=0} \qquad \mathrm{-3y^2-3x=0}$

$\mathrm{x^2-y=0} \qquad \mathrm{y^2+x=0}$

$\mathrm{y=x^2}$ ....(1) $\qquad \mathrm{x=-y^2}$........(2)

$\mathrm{y^2=x^4}$ ..... (3)

$\mathrm{x^4+x=0}$ sub (2)

$\mathrm{x(x^3+1)=0}$

$\mathrm{x=0;\ x^3+1=0}$ and $\mathrm{x=-1}$

From (3) y=0, ±1

The stationary points are (0,0), (-1,1)

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5. Examine $f(x,y)=x^3+y^3-12x-3y+20$ for its extreme values.

$$Given \quad f(x,y)=x^3+y^3-12x-3y+20$$

$$f_x=3x^2-12 \quad \& \quad f_y=3y^2-3$$

$$Consider \ f_x=0 \quad \& \quad f_y=0$$

$$3x^2-12=0 \quad \& \quad 3y^2-3=0$$

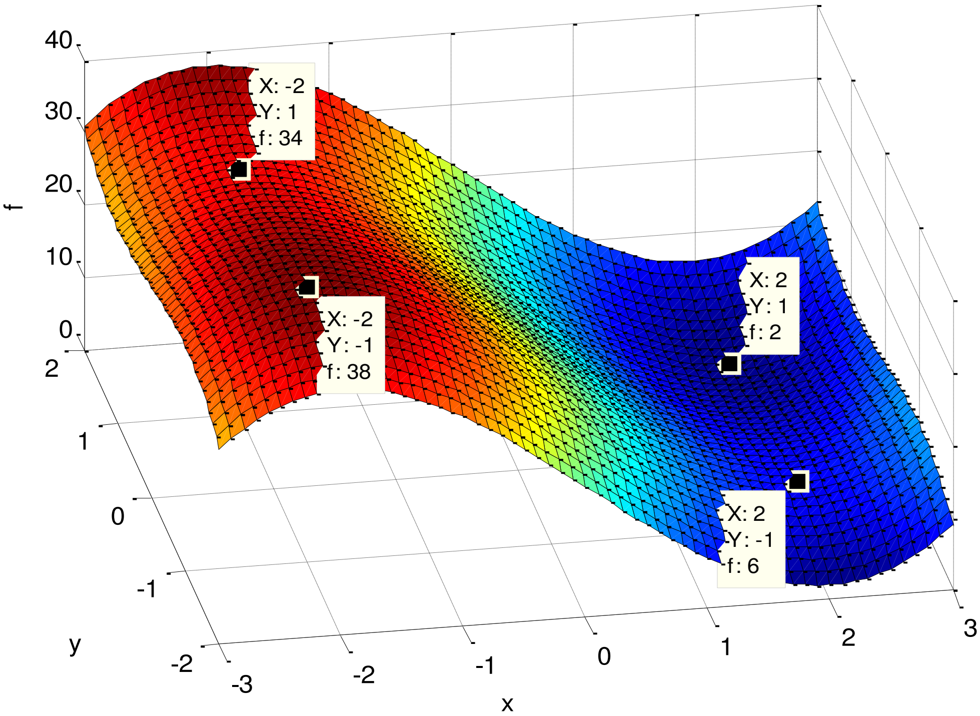
$$x^2=4 \quad \& \quad y^2=1$$

$$x=\pm2 \quad \& \quad y=\pm1$$

Therefore stationary points are (2,1), (2,–1), (–2, –1) and (–2,1)

$$f_{xx}=6x \qquad f_{yy}=6y \qquad f_{xy=0}$$

Point	$A=f_{xx}$	$B=f_{xy}=0$	$C=f_{yy}$	$AC-B^2$	**Nature**	**Extreme value**
(2,1)	12>0	0	6	72 > 0	$AC-B^2>0$ $A>0$ Minimum	$f(2,1)=$ $2^3+1^3-12\times2-3+20$ $=2$
(2, –1)	12	0	–6	–72 < 0	Saddle point	neither maximum nor minimum
(–2,–1)	–12<0	0	–6	72 > 0	$AC-B^2>0$ $A<0$ Maximum	$f(-2,-1)=$ $-2^3-1^3+12\times2+3+20$ $=38$
(–2,1)	–12	0	6	–72 < 0	Saddle point	neither maximum nor minimum



6. Find the extreme values of the function $f(x,y)=x^3+y^3-3x-12y+20$.

Given: $f(x,y)=x^3+y^3-3x-12y+20$

$f_x(x,y)=3x^2-3$ $\qquad$ $f_y(x,y)=3y^2-12$

$A=f_{xx}(x,y)=6x$ $\qquad$ $B=f_{xy}(x,y)=0$ $\qquad$ $C=f_{yy}(x,y)=6y$

To find the stationary points.

$f_x=0$	$f_y=0$
$\therefore 3x^2-3=0$ $x^2-1=0$ $x=\pm1$	$\therefore 3y^2-12=0$ $y^2-4=0$ $y=\pm2$

The stationary points are (1,2), (1, –2), (–1,2), (–1, –2)

	(1, 2)	(1, –2)	(–1,2)	(–1,–2)
$A=6x$	6 > 0	6 > 0	–6 < 0	–6 < 0
B	0	0	0	0
$AC-B^2$	$36xy$	$36xy$	$36xy$	$36xy$
$AC-B^2$	72 > 0	–72 < 0	–72 < 0	72 > 0
Conclusion	Min. point	Saddle point	Saddle point	Max. point.

Maxima value of $f(x,y)$ is

$f(-1,-2)=(-1)^3+(-2)^3-3(-1)-12(-2)+20=38$

Minimum value of $f(x,y)$ is

$f(1,2)=(1)^3+(2)^3-3(1)-12(2)+20=2$

7. Find the maxima, minima of the function $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$

Given $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$

$f_x=\dfrac{\partial f}{\partial x}=4x^3-4x+4y$ $\qquad$ & $\qquad$ $f_y=\dfrac{\partial f}{\partial y}=4y^3+4x-4y$

$f_{xx}=\dfrac{\partial^2 f}{\partial x^2}=12x^2-4$, $\qquad$ $f_{xy}=\dfrac{\partial^2 f}{\partial x\partial y}=4$ $\qquad$ $f_{yy}=\dfrac{\partial^2 f}{\partial y^2}=12y^2-4$

Consider $f_x=0$ $\qquad$ & $\qquad$ $f_y=0$

$x^3-x+y=0$ $\qquad$ & $\qquad$ $y^3+x-y=0$

Adding, we get

$$x^3-x+y+y^3+x-y=0$$

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$$x^3 + y^3 = 0$$

$$y^3 = -x^3$$

$$y = -x$$

Substitute $y = -x$ in $f_x = 0$, we get

$$x^3 - x - x = 0$$

$$x^3 - 2x = 0$$

$$x(x^2 - 2) = 0$$

$$x = 0, x = \pm\sqrt{2}$$

Since $y = -x$, we get $y = 0,\ y = \mp\sqrt{2}$

$\therefore$ The critical points are $(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

At $(-\sqrt{2}, \sqrt{2})$

$A = f_{xx} = 12(-\sqrt{2})^2 - 4 = 24 - 4 = 20 > 0$

$C = f_{yy} = 12(\sqrt{2})^2 - 4 = 24 - 4 = 20 > 0$

$B = f_{xy} = 4$

$\therefore AC - B^2 = (20)(20) - 4^2 = 384 > 0$

$\therefore (-\sqrt{2}, \sqrt{2})$ is a point of minimum value.

The minimum value is

$$f\left(-\sqrt{2}, \sqrt{2}\right) = \left(-\sqrt{2}\right)^4 + \left(\sqrt{2}\right)^4 - 2\left(-\sqrt{2}\right)^2 + 4\left(-\sqrt{2}\right)\left(\sqrt{2}\right) - 2\left(\sqrt{2}\right)^2$$

$$= 4 + 4 - 4 - 8 - 4 = -8$$

At $(\sqrt{2}, -\sqrt{2})$

$A = f_{xx} = 12(\sqrt{2})^2 - 4 = 24 - 4 = 20 > 0$

$C = f_{yy} = 12(-\sqrt{2})^2 - 4 = 24 - 4 = 20 > 0$

$B = f_{xy} = 4$

$\therefore AC - B^2 = f_{xx}f_{yy} - f_{xy}^2 = (20)(20) - 4^2 = 384 > 0$

$\therefore (\sqrt{2}, -\sqrt{2})$ is a point of minimum value.

The minimum value is

$$f\left(\sqrt{2}, -\sqrt{2}\right) = \left(\sqrt{2}\right)^4 + \left(-\sqrt{2}\right)^4 - 2\left(\sqrt{2}\right)^2 + 4\left(\sqrt{2}\right)\left(-\sqrt{2}\right) - 2\left(-\sqrt{2}\right)^2$$

$$= 4 + 4 - 4 - 8 - 4 = -8$$

At (0, 0)

$$A = f_{xx} = 12(0) - 4 = -4 < 0$$

$$C = f_{yy} = 12(0) - 4 = -4 < 0$$

$$B = f_{xy} = 4$$

$$\therefore AC - B^2 = f_{xx}f_{yy} - f_{xy}^2 = (-4)(-4) - 4^2 = 16 - 16 = 0$$

$\therefore$ (0, 0) cannot be a extreme point. It is a saddle point.

8. Investigate for the maxima and minima if any of the function $f(x,y)=x^3+y^3-3xy$

$$Given: f(x,y)=x^3+y^3-3xy$$

$f_x=0$	$f_y=0$
$3x^2-3y=0$ $3x^2=3y$ $x^2=y$ ...(1)	$3y^2-3x=0$ $3y^2=3x$ $y^2=x$ ..(2)

To find stationary points:

$(1)\Rightarrow\ y=x^2$ .....(3)

$(2)\Rightarrow y^2=x$ .....(4)

$(3)\Rightarrow x^4=y^2$ .....(5)

Substituting (4) in (5), we get

$$x^4=x$$

$$x^4-x=0$$

$$x(x^3-1)=0$$

$$\Rightarrow\ x=0 \text{ or } 1$$

Put x = 0 in (3), y = 0

x = 1 in (3), y = 1

Therefore the stationary points are (0, 0), (1, 1)

	At (0, 0)	At (1, 1)
$A=\dfrac{\partial^2 f}{\partial x^2}=6x$	0	0
$B=\dfrac{\partial^2 f}{\partial x\partial y}=-3$	–3	–3
$C=\dfrac{\partial^2 f}{\partial y^2}=6y$	0	6
$AC-B^2$	–9 < 0	36–9 = 27 > 0
Result:	$A=0$ $AC-B^2>0$ No extremum value The method fails	$A>0$ $AC-B^2>0$ point of maximum value

Maximum value is $f(1,1)=1^3+1^3-3=-1$

9. Test for maxima and minima of the function $f(x,y)=x^3y^2(6-x-y)$

Given

$$f(x,y)=x^3y^2(6-x-y)$$
$$=6x^3y^2-x^4y^2-x^3y^3$$

$$f_x(x,y)=18x^2y^2-4x^3y^2-3x^2y^3$$

$$f_y(x,y)=12x^3y-2x^4y-3x^3y^2$$

$$A=f_{xx}(x,y)=36xy^2-12x^2y^2-6xy^3$$

$$B=f_{xy}(x,y)=36x^2y-8x^3y-9x^2y^2$$

$$C=f_{yy}(x,y)=12x^3-2x^4-6x^3y$$

To find the stationary points.

$f_x = 0$	$f_y = 0$
$18x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ $x^2y^2(18 - 4x - 3y) = 0$ $x = 0$ $y = 0$ $4x + 3y = 18$ $when\ x = 0,\ y = -6$ $when\ y = 0,\ x = \frac{9}{2}$	$12x^3y - 2x^4y - 3x^3y^2 = 0$ $x^3y(12 - 2x - 3y) = 0$ $x = 0$ $y = 0$ $2x + 3y = 12$ $when\ x = 0,\ y = 4$ $when\ y = 0,\ x = 6$

$4x + 3y = 18 - - - - - (1)$ $\quad$ $2x + 3y = 12 - - - - - (2)$

(1) – (2) $\Rightarrow 2x = 6;\ x = 3$

Substitute in (2), $2(3) + 3y = 12;\ 3y = 6;\ y = 2$

$\therefore$ The stationary points are $(0,0)$, $(0,-6)$, $(0,4)$, $\left(\frac{9}{2}, 0\right)$, $(6,0)$, $(3,2)$

	$(0,0)$	$(0,-6)$	$\left(\frac{9}{2},0\right)$	$(6,0)$	$(3,2)$
A	0	0	0	0	-144
B	0	0	0	0	-108
C	0	0	$\frac{8019}{8}$	0	-162
$AC - B^2$	0	0	0	0	11664
Decision	Inconclusive	Inconclusive	Inconclusive	Inconclusive	Max. point

Thus $(3,2)$ is a maximum point.

$\therefore$ The maximum value is $f(3,2) = x^3y^2(6 - x - y) = 108$

10. Find the maximum and minimum values of $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f_x = 3x^2 + 3y^2 - 30x + 72$

$f_y = 6xy - 30y$

$A = f_{xx} = 6x - 30$

$C = f_{yy} = 6x - 30$

$B = f_{xy} = 6y$

$Let\ \ f_x = 0$ & $f_y = 0$

$$3x^2 + 3y^2 - 30x + 72 = 0 \qquad 6xy - 30y = 0$$

$$x^2 + y^2 - 10x + 24 = 0 ....(1) \qquad y(x-5) = 0$$

$$y = 0 \ \ or \ \ x = 5$$

$When\ \ y = 0,\ (1)\ gives$ $\qquad$ $When\ x = 5,\ \ (1)\ \ gives$

$$x^2 - 10x + 24 = 0 \qquad 5^2 + y^2 - 10(5) + 24 = 0$$

$$(x-6)(x-4) = 0 \qquad y^2 = 1$$

$$x = 6,\ x = 4 \qquad y = \pm 1$$

$\therefore$ the stationary points are $(4,0),\ (6,0),\ (5,1),\ (5,-1).$

Point	$A = 6x - 30$	$B = 6y$	$C = 6x - 30$	$\mathrm{AC - B^2}$	Nature
(4, 0)	– 6 < 0	0	–6	36 > 0	Maximum
(6, 0)	6 > 0	0	6	6 > 0	Minimum
(5, 1)	0	6	0	–36 < 0	Saddle point
(5,–1)	0	–6	0	–36 < 0	Saddle point

The maximum value of the function $f(x,y)$ at (4,0) is

$$\begin{aligned} \mathrm{f(4,0)} &= \mathrm{x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x} \\ &= 64 - 240 + 288 \\ &= 112 \end{aligned}$$

The minimum value of the function $f(x,y)$ at (6,0) is

$$\begin{aligned} \mathrm{f(6,0)} &= \mathrm{x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x} \\ &= 216 - 540 + 432 \\ &= 108 \end{aligned}$$

11. Discuss the maxima and minima of $f(x,y) = x^3y^2\left(1 - x - y\right)$.

Given $f(x,y) = x^3\ y^2\ (1 - x - y)$

$$= x^3\ y^2 - x^4\ y^2 - x^3\ y^3$$

$$f_x(x,y) = 3x^2\ y^2 - 4x^3\ y^2 - 3x^2\ y^3$$

$$f_y(x,y) = 2x^3\ y - 2x^4\ y - 3x^3\ y^2$$

$$A = f_{xx}(x,y) = 6x\,y^2 - 12x^2\ y^2 - 6x\,y^3$$

$$B = f_{xy}(x,y) = 6x^2y - 8x^3\ y - 9x^2y^2$$

$$C = f_{yy}(x,y) = 2x^3 - 2x^4 - 6x^3y$$

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To find the stationary points.

$f_x = 0$	$f_y = 0$
$3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0$ $x^2 y^2[3-4x-3y]=0$ $x=0$ $y=0$ $4x+3y=3$ $when\ x=0,\ y=1$ $when\ y=0,\ x=\frac{3}{4}$	$2x^3 y - 2x^4 y - 3x^3 y^2 = 0$ $x^3 y[2-2x-3y]=0$ $x=0$ $y=0$ $2x+3y=2$ $when\ x=0,\ y=\frac{2}{3}$ $when\ y=0,\ x=1$

$$4x+3y=3 \qquad ...(1)$$
$$2x+3y=2 \qquad ...(2)$$
$$(1)-(2) \Rightarrow\ 2x=1; \qquad x=\frac{1}{2}$$
$$(1)-2\times(2) \Rightarrow\ -3y=-1; \qquad y=\frac{1}{3}$$

$\therefore$ The stationary points are $(0,0), \left(\frac{1}{2},\frac{1}{3}\right), (0,1), \left(0,\frac{2}{3}\right), \left(\frac{3}{4},0\right)$ and $(1,0)$

	$(0,0)$	$\left(\frac{1}{2},\frac{1}{3}\right)$	$(0,1)$	$\left(0,\frac{2}{3}\right)$	$\left(\frac{3}{4},0\right)$	$(1,0)$
A	0	$-\frac{1}{9}<0$	0	0	0	0
B	0	$-\frac{1}{12}$	0	0	$\frac{27}{128}$	0
C	0	$-\frac{1}{8}$	0	0	0	0
AC−B²	0	$\frac{1}{144}>0$	0	0	0	0
	Inconclusive	Max. point	Inconclusive	Inconclusive	Inconclusive	Inconclusive

Thus $\left(\frac{1}{2},\frac{1}{3}\right)$ is a maximum point.

$\therefore$ The maximum value $f(x,y) = f\left(\frac{1}{2},\frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left[1-\frac{1}{2}-\frac{1}{3}\right] = \frac{1}{432}$

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12. Discuss the maxima and minima of $f(x,y)=x^2+xy+y^2+\frac{1}{x}+\frac{1}{y}$

Given: $f(x,y)=x^2+xy+y^2+\frac{1}{x}+\frac{1}{y}$

Differentiate partially w.r.t $x \text{ and } y$.

$$f_x=2x+y-\frac{1}{x^2}$$

$$f_y=2y+x-\frac{1}{y^2}$$

$$A=f_{xx}=2+\frac{2}{x^3}$$

$$B=f_{xy}=1$$

$$C=f_{yy}=2+\frac{2}{y^3}$$

Equating $f_x=0 \text{ and } f_y=0$ to find the stationary points

$$2x+y-\frac{1}{x^2}=0....(1)$$

$$2y+x-\frac{1}{y^2}=0...(2)$$

Subtracting (1) & (2)

$$2x+y-\frac{1}{x^2}-2y-x+\frac{1}{y^2}=0.$$

$$x-y-\frac{1}{x^2}+\frac{1}{y^2}=0.$$

$$x-y+\frac{1}{y^2}-\frac{1}{x^2}=0.$$

$$x-y+\frac{x^2-y^2}{x^2y^2}=0.$$

$$x^2y^2(x-y)+x^2-y^2=0$$

$$x^2y^2(x-y)+(x-y)(x+y)=0$$

$$x^2y^2(x-y)+(x-y)(x+y)=0$$

$$(x-y)(x^2y^2+x+y)=0$$

$$(x-y)=0 \text{ or } (x^2y^2+x+y)=0$$

consider $(x-y)=0 \Rightarrow x=y$ ....(3)

Substitute (3) in (1)

$$2y+y-\frac{1}{y^2}=0$$

$$3y-\frac{1}{y^2}=0$$

$$\frac{3y^3-1}{y^2}=0$$

$$3y^3-1=0$$

$$3y^3=1$$

$$y^3=\frac{1}{3} \qquad y=\left(\frac{1}{3}\right)^{\frac{1}{3}}$$

Since $x=y,$ the stationary point is $\left[\left(\frac{1}{3}\right)^{\frac{1}{3}},\left(\frac{1}{3}\right)^{\frac{1}{3}}\right]$.

	At $\left\{\left(\frac{1}{3}\right)^{\frac{1}{3}},\left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}$
$A=f_{xx}=2+\frac{2}{x^3}$	$A=2+\frac{2}{\left(\frac{1}{3}\right)}=8$
$B=f_{xy}=1$	B = 1
$C=f_{yy}=2+\frac{2}{y^3}$	$C=2+\frac{2}{\frac{1}{3}}=8$
$AC-B^2$	64 – 1 = 63 > 0

Since $AC-B^2>0$ & $A>0,$ $f(x,y)$ has minimum at $\left\{\left(\frac{1}{3}\right)^{\frac{1}{3}},\left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}$

The minimum value of $f\left(\left(\frac{1}{3}\right)^{\frac{1}{3}},\left(\frac{1}{3}\right)^{\frac{1}{3}}\right)=\left(\frac{1}{3}\right)^{\frac{2}{3}}+\left(\frac{1}{3}\right)^{\frac{1}{3}}\left(\frac{1}{3}\right)^{\frac{1}{3}}+\left(\frac{1}{3}\right)^{\frac{2}{3}}+\frac{1}{\left(\frac{1}{3}\right)^{\frac{1}{3}}}+\frac{1}{\left(\frac{1}{3}\right)^{\frac{1}{3}}}$

$$=2\left(\frac{1}{3}\right)^{\frac{2}{3}}+\left(\frac{1}{9}\right)^{\frac{1}{3}}+\frac{2}{\left(\frac{1}{3}\right)^{\frac{1}{3}}}$$

$$=2\left(\frac{1}{3}\right)^{\frac{2}{3}}+\left(\frac{1}{9}\right)^{\frac{1}{3}}+2(3)^{\frac{1}{3}}$$

$$=2\left(\frac{1}{9}\right)^{\frac{1}{3}}+\left(\frac{1}{9}\right)^{\frac{1}{3}}+2(3)^{\frac{1}{3}}$$

$$=3\left(\frac{1}{9}\right)^{\frac{1}{3}}+2(3)^{\frac{1}{3}}$$

$$=(3)^{\frac{1}{3}}+2(3)^{\frac{1}{3}}$$

$$=3(3)^{\frac{1}{3}}$$

$$=(3)^{\frac{4}{3}}$$

The minimum value of $f\left(\left(\frac{1}{3}\right)^{\frac{1}{3}},\left(\frac{1}{3}\right)^{\frac{1}{3}}\right)=3^{\frac{4}{3}}$

Constrained Maxima and Minima

Sometimes we need to find the extremum of $f(x,y,z)=0$ subject to a condition of $\phi(x,y,z)=0$. The extremum in such case is said to be constrained maxima and minima.

Now one variable, say z, may be obtained from $\phi(x,y,z)=0$ and it may be substituted in $f(x,y,z)=0$. The resulting function is a function of two variables and the usual method can be applied to find the extremum values. If this is complicated we can apply the following method.

Lagrange's Method

Suppose we need to find the extremum of $f(x,y,z)=0$ subject to a condition of $\phi(x,y,z)=0$.
Working Rule:

1. Write the auxiliary function $g=f+\lambda\phi$, where λ is the Lagrange multiplier.

2. Differentiate partially g w.r.t. $x,\ y,\ z\ and\ \lambda$

3. Solve the equations $g_x=0,\ g_y=0,\ g_z=0,\ and\ \ g_\lambda=0$ to get the stationary points.

4. At this stationary point , extremum exists.

Note: Lagrange's method does not indicate whether the extremum is maxima or minima. It is decided by the physical condition of the problem.

1. Divide 24 into three parts such that the continued product of the first, square of the second and cube of the third may be maximum.

Let $x,\ y,\ z$ be the three parts of the number 24. Then $x+y+z=24$.

Now we have to maximize $f(x,\ y,\ z)=xy^2z^3$ subject to the condition

$\phi(x,y,z)=x+y+z-24$.

Consider the auxiliary function $g=f(x,\ y,\ z)+\lambda\phi(x,y,z)$

$$g=\left(xy^2z^3\right)+\lambda(x+y+z-24)$$

Differentiate g w.r.t. $x,\ y,\ z,\lambda$, we get

$$g_x=\left(y^2z^3\right)+\lambda$$

$$g_y=\left(2xyz^3\right)+\lambda$$

$$g_z=\left(3xy^2z^2\right)+\lambda$$

$$g_\lambda=x+y+z-24$$

Consider $g_x=g_y=g_z=g_\lambda=0$, we get

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$$\left(y^2z^3\right)=-\lambda.........(1)$$

$$\left(2xyz^3\right)=-\lambda.......(2)$$

$$\left(3xy^2z^2\right)=-\lambda.....(3)$$

$$x+y+z=24......(4)$$

Equating, we have $\left(y^2z^3\right)=\left(2xyz^3\right)=\left(3xy^2z^2\right)$

$$\left(y^2z^3\right)=\left(2xyz^3\right)=\left(3xy^2z^2\right)$$

$$yz=2xz=3xy$$

$$\frac{1}{x}=\frac{2}{y}=\frac{3}{z}=\frac{1}{k}$$

$$i.e.\ \ x=k,\ y=2k,\ z=3k$$

Substitute these values in (4)

$$k+2k+3k=24$$

$$6k=24$$

$$k=4$$

Therefore $x=4\quad y=8,\quad z=12$

2. Find the minimum distance from the origin to the surface $z^2=1+xy$.

Let $P(x,\ y,\ z)$ be a point on the curve. The distance from the origin to this point is

$$d=\sqrt{(x-0)^2+(y-0)^2+(y-0)^2}$$

$$d^2=(x-0)^2+(y-0)^2+(y-0)^2$$

$$Let\ \ f(x,y,z)=x^2+y^2+z^2$$

Now we have to maximize f subject to the curve $z^2=xy+1$

Therefore

$$f(x,y,z)=x^2+y^2+z^2=x^2+y^2+xy+1$$

$$f(x,y)=x^2+y^2+xy+1$$

Differentiate f w.r.t $x,\ y$, we get

$$f_x=2x+y$$

$$f_y=2y+x$$

Solving $f_x=0\ and\ f_y=0$, we have

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$$2x + y = 0.....(1)$$

$$2y + x = 0$$

$$Subtracting,\ x - y = 0 \quad i.e. \quad x = y$$

From (1), $3x = 0,\ i.e.\ x = 0$ and hence $y = 0$ and hence $(0,0)$ is the stationary point.

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 1$

$\therefore\ at\ (0,0)\ \ A = f_{xx} = 2, \quad B = f_{xy} = 1, \quad C = f_{yy} = 2$

$Also\ \Delta = AC - B^2 = 4 - 1 = 3 > 0 \quad and \quad A = 2 > 0$

Hence f has minimum at $x = 0$ & $y = 0$

But $z^2 = 1 + xy$ gives $z^2 = 1\ and\ z = \pm 1$.

Therefore the required points on the surfaces which gives the minimum distances are $A(0,0,1)\ and\ B(0,0,-1)$.

Another Method:

Consider the auxiliary function $g = f(x,\ y,\ z) + \lambda\phi(x, y, z)$

$$g = \left(x^2 + y^2 + z^2\right) + \lambda\left(z^2 - xy - 1\right)$$

Differentiate g w.r.t. $x,\ y,\ z\ and\ \lambda$, we get

$$g_x = (2x) + \lambda(-y)$$

$$g_y = (2y) + \lambda(-x)$$

$$g_z = (2z) + \lambda(2z)$$

$$g_\lambda = \left(z^2 - xy - 1\right)$$

Consider $g_x = g_y = g_z = g_\lambda = 0$, we get

$$0 = (2x) + \lambda(-y) \quad i.e. \quad \lambda = \frac{2x}{y}$$

$$0 = (2y) + \lambda(-x) \quad i.e. \quad \lambda = \frac{2y}{x}$$

$$0 = (2z) + \lambda(2z) \quad i.e. \quad \lambda = -1$$

$$0 = \left(z^2 - xy - 1\right) \quad i.e. \quad z^2 = xy + 1$$

Equating, the first two, $\dfrac{2x}{y} = \dfrac{2y}{x}$

$$2x^2 = 2y^2$$

$$x^2 = y^2$$

$$x = y$$

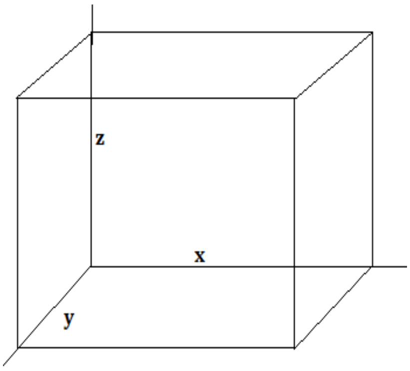
$$\therefore 2x - \lambda x = 0$$

$$i.e.\ \ x = 0 \quad and \quad y = 0$$

$$\text{Sin}\,ce\ z^2 = 1 + xy,\ \ z^2 = 1,\ \ z = \pm 1$$

Hence the stationary points are $A(0,0,1)$ *and* $B(0,0,-1)$

3. A rectangular box open at the top, is to have a capacity of 108 cu.ms. Find the dimensions of the box requiring the least material for its construction.



Let x, y, z be the length, breadth and height of the box.

Surface area $= xy + 2yz + 2xz$

(should be minimum)

Volume $= xyz = 108$

Let the auxiliary function F be

$$F(x,y,z) = (xy + 2yz + 2zx) + \lambda(xyz - 108)$$

Where λ is Lagrange multiplier.

Differentiate $F \quad w.r.t. \quad x, y, z$, we get

$$F_x = \frac{\partial F}{\partial x} = y + 2z + \lambda yz;$$

$$F_y = \frac{\partial F}{\partial y} = x + 2z + \lambda xz;$$

$$F_z = \frac{\partial F}{\partial z} = 2x + 2y + \lambda xy$$

F is extremum when

$$F_x = 0 \Rightarrow y + 2z + \lambda yz = 0 \Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda \qquad ...(2)$$

$$F_y = 0 \Rightarrow x + 2z + \lambda xz = 0 \Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda \qquad ...(3)$$

$$F_z = 0 \Rightarrow 2x + 2z + \lambda xy = 0 \Rightarrow \frac{2}{y} + \frac{2}{x} = -\lambda \qquad ...(4)$$

Equating first two,

we get

$$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x}$$

$$x = y$$

Equating the last two,

we get

$$\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$$

$$\frac{1}{z} = \frac{2}{y}$$

$$y = 2z$$

Therefore, we get x = y = 2z

But Volume is xyz = 108

$$(2z)(2z)z = 108$$

$$4z^3 = 108$$

$$z^3 = \frac{108}{4} = 27$$

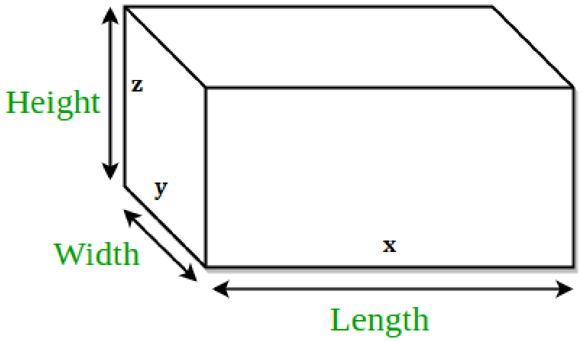
z = 3

$$\therefore x = 6,\ \ y = 6,\ \ z = 3$$

Cost is minimum when x = 6, y = 6, z = 3.

Thus the dimensions of the box are 6, 6, 3.

4. Find the dimensions of the rectangular box open at the top of maximum capacity whose surface area is 432 sq cm.



Let x, y, z be the length, breadth and height of the box.

Given Surface area $xy+2yz+2xz=432$

Volume $V=xyz$ (should be maximum)

Let the auxiliary function F be

$$F=xyz+\lambda(xy+2yz+2xz-432)$$

Where λ is Lagrange multiplier.

Differentiate F $w.r.t.$ x, y, z, we get

$$F_x=yz+\lambda(y+2z)$$

$$F_y=xz+\lambda(x+2z)$$

$$F_z=xy+\lambda(2y+2x)$$

Consider $F_x=F_y=F_z=0$

$$0=yz+\lambda(y+2z) \quad i.e. \quad -\frac{1}{\lambda}=\frac{y+2z}{yz}=\frac{1}{z}+\frac{2}{y}$$

$$0=xz+\lambda(x+2z) \quad i.e. \quad -\frac{1}{\lambda}=\frac{x+2z}{xz}=\frac{1}{z}+\frac{2}{x}$$

$$0=xy+\lambda(2y+2x) \quad i.e. \quad -\frac{1}{\lambda}=\frac{2y+2x}{xy}=\frac{2}{x}+\frac{2}{y}$$

From the first two, we have

$$\frac{1}{z}+\frac{2}{y}=\frac{1}{z}+\frac{2}{x}$$

$$\frac{2}{y}=\frac{2}{x}$$

$$x=y$$

From the last two, we have

$$\frac{1}{z}+\frac{2}{x}=\frac{2}{y}+\frac{2}{x}$$

$$\frac{1}{z}=\frac{2}{y}$$

$$y=2z$$

$\therefore$ we get $x=y=2z$

But surface area $xy+2yz+2xz=432$

$$2z.2z+2.2z.z+2.2z.z=432$$

$$12z^2=432$$

$$z^2=36$$

$$z=6$$

$$\therefore x=12,\ y=12,\ z=6$$

$$\therefore Max\ volume=12\times12\times6=864\ units$$

5. Find the maximum value of $x^m y^n z^p$, when $x+y+z=a$.

Let $f = x^m y^n z^p$ and $\phi = x+y+z-a$

Therefore the auxiliary function is $g = f + \lambda\phi = x^m y^n z^p + \lambda(x+y+z-a)$

The maximum values exists at $g_x = g_y = g_z = g_\lambda = 0$

$g_x = 0,$	$g_y = 0,$	$g_z = 0,$	$g_\lambda = 0$
$mx^{m-1} y^n z^p + \lambda = 0$	$nx^m y^{n-1} z^p + \lambda = 0$	$px^m y^n z^{p-1} + \lambda = 0$	$x+y+z-a=0$
$mx^{m-1} y^n z^p = -\lambda$	$nx^m y^{n-1} z^p = -\lambda$	$px^m y^n z^{p-1} = -\lambda$	$x+y+z=a$

From the above

$$-\lambda = mx^{m-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

Divide by $x^m y^n z^p$

$$i.e., \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}$$

Equating each ratio with the last ratio, we have

$$x = \frac{am}{m+n+p},\ y = \frac{an}{m+n+p},\ z = \frac{ap}{m+n+p}$$

Max. Value $f = x^m y^n z^p$

$$f = \frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}$$

6. Find the volume of the greatest rectangular parallelepiped inscribed in the ellipsoid whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Let a vertex of such parallelepiped by (x, y, z)

Then all other vertices will be $(\pm x, \pm y, \pm z)$

Then the sides of the solid be 2x, 2y, 2z (lengths)

Hence, the volume V = (2x) (2y) (2z) = 8xyz

Now, we have to maximize V subject to the condition $\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

Let $F = f + \lambda\phi = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

Differentiate F $w.r.t.$ x, y, z $and\ equate\ to\ 0,$ $we\ have$

$\frac{\partial F}{\partial x}=0 \Rightarrow\ 8yz+\frac{2x\lambda}{a^2}=0$ $8xyz=-\frac{2x^2\lambda}{a^2}$	$\frac{\partial F}{\partial y}=0 \Rightarrow\ 8xz+\frac{2y\lambda}{b^2}=0$ $8xyz=-\frac{2y^2\lambda}{b^2}$	$\frac{\partial F}{\partial z}=0 \Rightarrow\ 8xy+\frac{2z\lambda}{c^2}=0$ $8xyz=-\frac{2z^2\lambda}{c^2}$

Equating all the ratios,

$$\frac{2x^2\lambda}{a^2}=\frac{2y^2\lambda}{b^2}=\frac{2z^2\lambda}{c^2}$$

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

$$i.e., \frac{\frac{x^2}{a^2}}{1}=\frac{\frac{y^2}{b^2}}{1}=\frac{\frac{z^2}{c^2}}{1}=\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}{3}=\frac{1}{3}$$

$$\therefore\quad x^2=\frac{a^2}{3} \Rightarrow\ x=\frac{a}{\sqrt{3}} \qquad y^2=\frac{b^2}{3} \Rightarrow\ y=\frac{b}{\sqrt{3}} \qquad z^2=\frac{b^2}{3} \Rightarrow\ z=\frac{c}{\sqrt{3}}$$

The extremum point is $\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$

This will not give minimum V because when x = 0 , V = 0 when the solid becomes a rectangular sheet. Hence, this gives only maximum value.

$\therefore$ Maximum volume is $V=2x.2y.2z$

$$V=8\left(\frac{abc}{3\sqrt{3}}\right)$$

7. A rectangular box open at the top , is to have a volume of 32cc. Find the dimensions of the box which requires least material for its construction.

Let x, y, z be the length, breadth and height of the box.

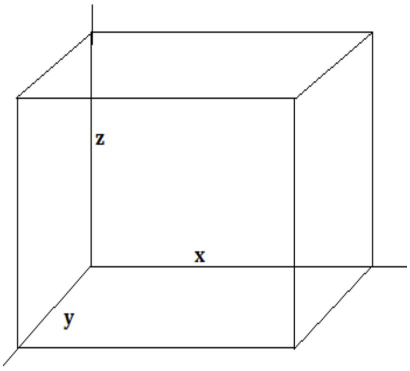
Surface area $=xy+2yz+2xz$ (minimised)

Volume $=xyz=32$

Let the auxiliary function F be

$F=(xy+2yz+2xz)+\lambda(xyz-32)$

Where λ is lagrange multiplier.



Differentiate F $w.r.t.$ x, y, z, we get

$$F_x = \frac{\partial F}{\partial x} = y + 2z + \lambda yz;$$

$$F_y = \frac{\partial F}{\partial y} = x + 2z + \lambda xz;$$

$$F_z = \frac{\partial F}{\partial z} = 2x + 2y + \lambda xy$$

F is extremum when

$$F_x = 0 \Rightarrow y + 2z + \lambda yz = 0 \Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda \qquad ...(2)$$

$$F_y = 0 \Rightarrow x + 2z + \lambda xz = 0 \Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda \qquad ...(3)$$

$$F_z = 0 \Rightarrow 2x + 2z + \lambda xy = 0 \Rightarrow \frac{2}{y} + \frac{2}{x} = -\lambda \qquad ...(4)$$

Equating first two,

we get

$$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x}$$

$$x = y$$

Equating the last two,

we get

$$\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$$

$$\frac{1}{z} = \frac{2}{y}$$

$$y = 2z$$

Therefore, we get x = y = 2z

But Volume = xyz = 32

$$2z.2z.z = 32$$

$$4z^3 = 32$$

$$z^3 = 8$$

$$z = 2$$

$$x = 4,\ y = 4,\ z = 2$$

$\therefore$ Material is minimum when the dimensions of the box are 4, 4, 2.

8. Using Lagrange's method, find the maximum of value of

$x^2 + y^2 + z^2$ *when* $x + y + z = 3a.$

Given $f = x^2 + y^2 + z^2 \quad and \quad \phi = x + y + z - 3a.$

Consider the lagrange's equation $g = f + \lambda\phi$

$$g = \left(x^2 + y^2 + z^2\right) + \lambda\left(x + y + z - 3a\right)$$

Differentiate 'g' w.r.t x, y, z *and* λ, and equate to 0. We get

$g_x = 2x + \lambda$

$g_x = 0$ *gives*

$2x + \lambda = 0$

$2x = -\lambda$

$g_y = 2y + \lambda$

$g_y = 0$ *gives*

$2y + \lambda = 0$

$2y = -\lambda$

$g_z = 2z + \lambda$

$g_z = 0$ *gives*

$2z + \lambda = 0$

$2z = -\lambda$

$g_\lambda = x + y + z - 3a$

$g_\lambda = 0$ *gives*

$x + y + z = 3a$

Equating, we have $2x = 2y = 2z \quad i.e.\ x = y = z.$

But $x + y + z = 3a$ *gives* $3x = 3a \quad i.e.\ x = a \quad and \quad hence \quad y = a,\ z = a.$

Therefore f has maximum at (a, a, a) and the maximum value is $f = a^2 + a^2 + a^2 = 3a^2$

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Exercise

1 Find the possible extreme point of $f(x,y)=x^2+y^2+\frac{2}{x}+\frac{2}{y}$.

2 Examine the following functions for extreme values

(i) $x^3y-3x^2-2y^2-4y-3$ (ii) $x^4+x^2y+y^2$ at (0,0)

3 Find the maximum and minimum values of $f(x,y)=\sin x\sin y\sin(x+y);\ 0<x,y<\pi.$

4 Show that, if the perimeter of a triangle is constant, its area is maximum when it is equilateral.

5 In a triangle ABC, find the maximum value of $\cos A\cos B\cos C$

6 Find the shortest and longest distances from the point (1,2,–1) to the sphere $x^2+y^2+z^2=24$

7 Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.

8 Find the minimum value of $x^2+y^2+z^2$ when $ax+by+cz=p$.

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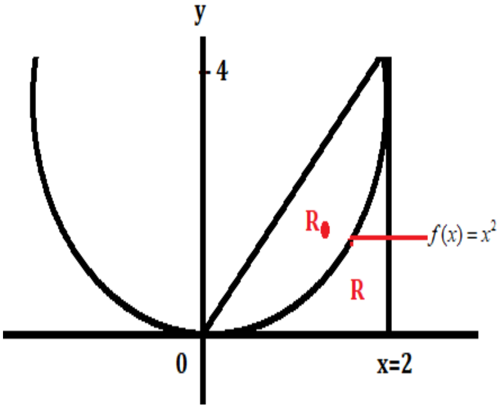
UNIT IV - INTEGRAL CALCULUS

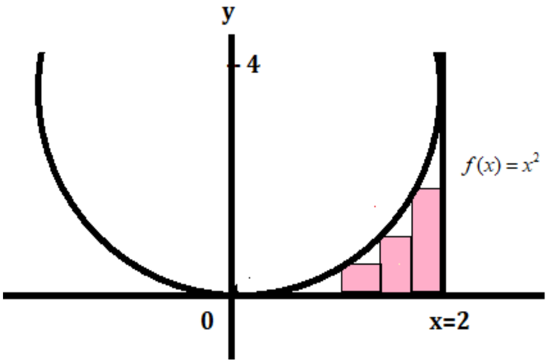
Introduction to Definite Integral

Consider the area R bounded by the $x-$ axis, the line $x=2$ and the curve $f(x)=x^2$ and the area of R_0. Since R and R_0 together comprise a triangle, whose area is $\frac{1}{2}bh=\frac{1}{2}2.4=4$.

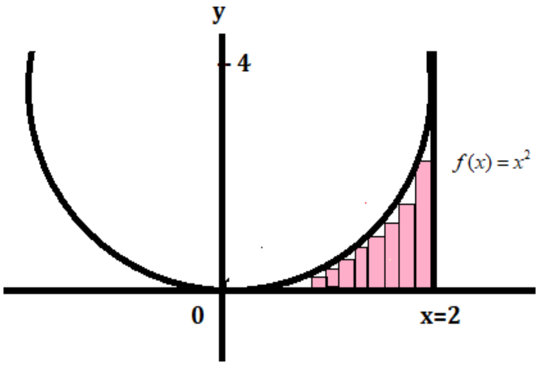
Hence finding the area of R is equivalent to finding the area of R_0.



Suppose we inscribe rectangles in the region R as shown in (A) and (B). Obviously the sum of areas of the rectangles is less than the area of R. If the width of the rectangles becomes smaller and smaller the sum of area of rectangles approaches the area of R. Thus the area of R is defined as a limit of the sum of areas of inscribed rectangles.



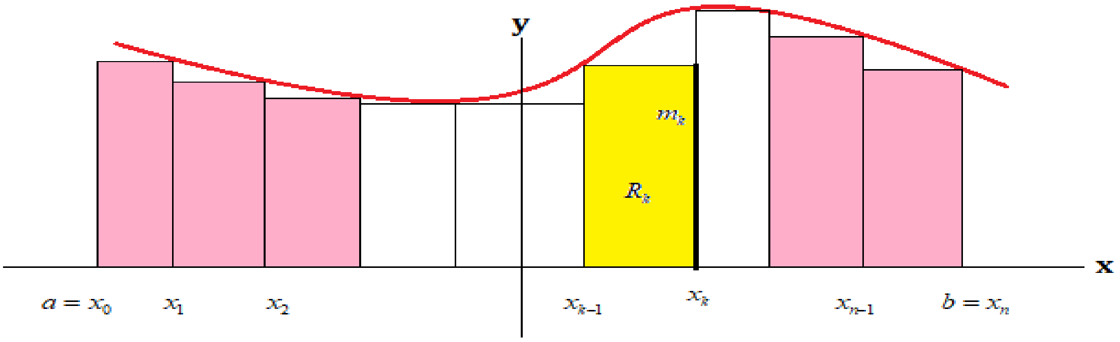
(A)



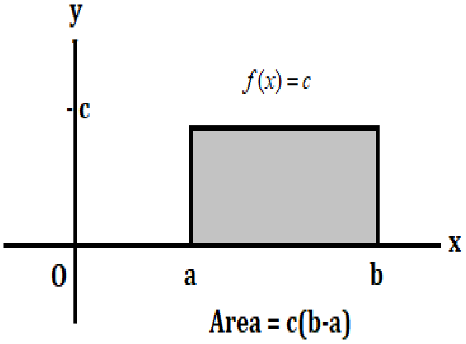
(B)

Consider a region bounded by a graph of non negative continuous function f on $[a,b]$, $x-$ axis, the line x $x=a$ and $x=b$. For any positive integer n divide $[a,b]$ into subintervals by introducing points of sub division $a=x_0$, x_1 , x_2 ,.....x_k , x_{k-1} , ...x_{n-1} , $b=x_n$. For each k between 1 and n the rectangle R_k has base $\left[x_{k-1},x_k\right]$ with length $\Delta x_k=x_k-x_{k-1}$ and has a height m_k. Hence the area of R_k is $m_k\Delta x_k$. Hence sum of all rectangles is $m_1\Delta x_1+m_2\Delta x_2+....+m_k\Delta x_k+......+m_n\Delta x_n$.

Note: $\Delta x_1+\Delta x_2+....+\Delta x_k+......+\Delta x_n=b-a$



Definition: Let f be continuous on $[a,b]$. The definite integral of f from a to b is $I \square m_1\Delta x_1 + m_2\Delta x_2 + ... + m_k\Delta x_k + ... + m_n\Delta x_n$ and it is denoted by $\int_a^b f(x)\,dx$.



Example: Let $f(x) = c$ for $a \le x \le b$. Show that $\int_a^b c\,dx = c(b-a)$

Since f assumes only the value c, for any partition $a = x_0 , x_1 , x_2 ,....x_k , x_{k-1} , ...x_{n-1} , b = x_n$ of $[a,b]$ and for any k between 1 and n, we have $m_1 = m_2 ... = m_k = ... = m_n = c$.

$$\therefore m_1\Delta x_1 + m_2\Delta x_2 + ... + m_n\Delta x_n = c\left(\Delta x_1 + \Delta x_2 + ... + \Delta x_n\right)$$

$$\int_a^b c\,dx = c(b-a)$$

DEFINITION OF A DEFINITE INTEGRAL (Riemann Integral (Integral as limit of sum)

If f is a function defined for $a \le x \le b$, we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = (b-a)/n$.

We let $x_0 (= a)$, $x_1, x_2, \ldots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots,$ x_n^* be any sample points in these subintervals, so x_i^* lies in the i th subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists. This sum is called Riemann sum of $f(x)$ corresponding to the partition. The integral is called Riemann integral of $f(x)$ on $[a,b]$. Also for the same partition there are many ways to choose x_i^* in the sub interval (x_{i-1}, x_i).

Note

1. $\int_a^b f(x)\,dx = \underset{n\to\infty}{Lt}\sum_{i=1}^{n} f\left(x_{i-1}\right)\left(x_i - x_{i-1}\right)$ is known as left end rule for evaluating Riemann integral.
2. $\int_a^b f(x)\,dx = \underset{n\to\infty}{Lt}\sum_{i=1}^{n} f\left(x_i\right)\left(x_i - x_{i-1}\right)$ is known as right end rule for evaluating Riemann integral.
3. $\int_a^b f(x)\,dx = \underset{n\to\infty}{Lt}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\left(x_i - x_{i-1}\right)$ is known as mid point rule for evaluating RI.

Note 1: The symbol dx simply indicates that the independent variable is x. The procedure of calculating an integral is called integration.

Note 2: The definite integral $\int_a^b f\,(x)dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f\,(x)dx = \int_a^b f\,(t)dt = \int_a^b f\,(r)dr$$

Note 3: If f is continuous on $[a,b]$, or if f has only a finite number of discontinuities, then f is integrable on $[a,b]$; that is, the definite integral $\int_a^b f\,(x)dx$ exists.

Theorem: If f is integrable on $[a,b]$, then

$\int_a^b f\,(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\,(x_i)\Delta x$ where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Example: Express $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$ as an integral on the interval $[0,\pi]$.

Comparing the given limit with the limit in above Theorem , we see that
$f(x) = x^3 + x\sin x$ and $a = 0, b = \pi$.

Therefore, $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x = \int_0^{\pi}(x^3 + x\sin x)dx$

Example : Evaluate $\int_0^3 x^3 - 6x\,dx$ using the Riemann sum corresponding to 6 sub intervals of equal length and applying (a) left end rule (b) right end rule.

(a) Here $a = 0,\ b = 3,\ n = 6,\ f(x) = 6x - x^3$ and interval width is $\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$

The left end points are $x_1 = 0, x_2 = 0.5,\ x_3 = 1,\ x_4 = 1.5,\ x_5 = 2$, and $x_6 = 2.5$. So the Riemann sum is

$$R_6 = \sum_{i=1}^{6} f\,(x_i)\Delta x$$

$$= f(0)\Delta x + f(0.5)\Delta x + f(1.0)\Delta x + f(1.5)\Delta x + f(2.0)\Delta x + f(2.5)\Delta x$$

$$= \frac{1}{2}(0 - 2.875 - 5 - 5.625 - 4 + 0.625)$$

$$= -8.43375$$

(b) The right end points are $x_1 = 0.5, x_2 = 1.0, x_3 = 1.5, x_4 = 2.0, x_5 = 2.5$, and $x_6 = 3.0$. So the Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x$$

$$= f(0.5)\Delta x + f(1.0)\Delta x + f(1.5)\Delta x + f(2.0)\Delta x + f(2.5)\Delta x + f(3.0)\Delta x$$

$$= \frac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9)$$

$$= -3.9375$$

Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the rectangles (above the x-axis) minus the sum of the areas of the rectangles (below the x-axis) .

Example : Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x} dx$.

The endpoints of the five subintervals are 1, 1.2, 1.4, 1.6, 1.8, and 2.0, so the midpoints are 1.1, 1.3, 1.5, 1.7, and 1.9. The width of the subintervals is

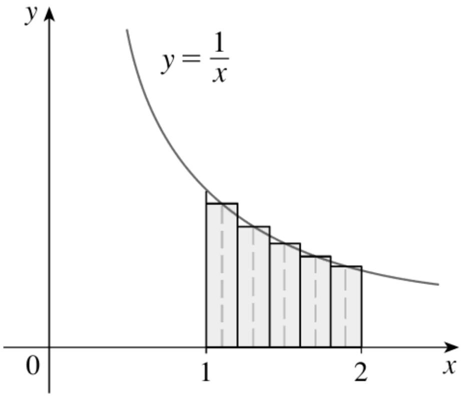
$$\Delta x = \frac{2-1}{5} = \frac{1}{5}$$

so the Midpoint Rule gives

$$\int_1^2 \frac{1}{x} dx = \Delta x[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$= 0.691908$$



Since $f(x) = 1/x > 0$ for $1 \le x \le 2$, the integral represents an area, and the approximation given by the Midpoint Rule is the sum of the areas of the rectangles.

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EVALUATING INTEGRALS (As a limit of the sum):

When we use a limit to evaluate a definite integral we need to know how to work with sums. The following equations give formulas for sums of powers of positive integers.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}; \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}; \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} K = nK; \qquad \sum_{i=1}^{n} Ka_i = K\sum_{i=1}^{n} a_i; \qquad \sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n}(a_i) \pm \sum_{i=1}^{n}(b_i)$$

Evaluate $\int_0^3 (x^3 - 6x)dx$ as the limit of the sum.

With n subintervals we have

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$

Thus $x_0 = 0, x_1 = 3/n, x_2 = 6/n, x_3 = 9/n$, and, in general, $x_j = 3i/n$. Since we are using right endpoints,

$$\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\,(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \underset{n\to\infty}{Lt}\ \frac{3}{n}\sum_{i=1}^{n}\left[\frac{27}{n^3}(i)^3 - \frac{18}{n}(i)\right]$$

$$= \underset{n\to\infty}{Lt}\ \frac{3}{n}\frac{27}{n^3}\sum_{i=1}^{n}(i)^3 - \underset{n\to\infty}{Lt}\ \frac{3}{n}\frac{18}{n}\sum_{i=1}^{n} i$$

$$= \underset{n\to\infty}{Lt}\ \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \underset{n\to\infty}{Lt}\ \frac{54}{n^2}\frac{n(n+1)}{2}$$

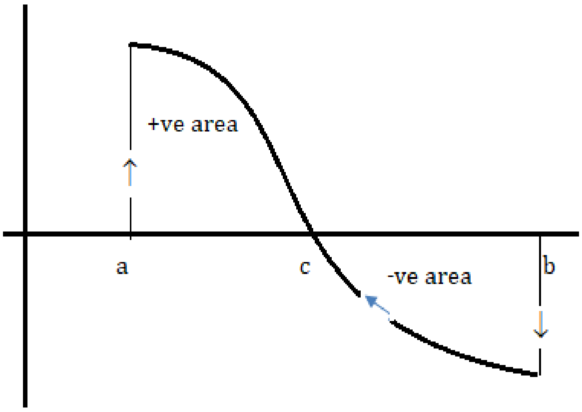
$$= \underset{n\to\infty}{Lt}\ \frac{81}{4}\left(1+\frac{1}{n}\right)^2 - \underset{n\to\infty}{Lt}\ \frac{54}{2}\left(1+\frac{1}{n}\right)$$

$$= \frac{81}{4} - \frac{54}{2} = -6.75$$

This integral can't be interpreted as an area because f takes on both positive and negative values. But it can be interpreted as the difference of areas $A_1 - A_2$, where A_1 and A_2

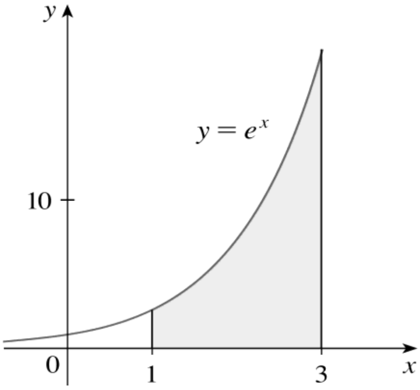
$$\int_0^3 (x^3 - 6x)dx = A_1 - A_2 = -6.75$$

Note: An area whose boundary is described in the anti-clockwise direction is considered positive area and an area whose boundary is described in the clockwise direction is taken as negative.



Example : Set up an expression for $\int_1^3 e^x\, dx$ as a limit of sums.

Here we have $f(x) = e^X, a = 1, b = 3,$ and $\Delta x = \frac{b-a}{n} = \frac{2}{n}$



So $x_0 = 1, x_1 = 1 + 2/n, x_2 = 1 + 4/n, x_3 = 1 + 6/n,$ and $x_i = 1 + \frac{2i}{n}$

We get,

$$\int_1^3 e^x\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\,(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty}\ \frac{2}{\mathrm{n}}\ \sum_{i=1}^{n} e^{\left(1+\frac{2i}{n}\right)}$$

PROPERTIES OF THE DEFINITE INTEGRAL

1. $\int_b^a f\,(x)dx = -\int_a^b f\,(x)dx$

2. $\int_a^a f\,(x)dx = 0$

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3.$\int_a^b c\,dx = c(b-a)$, where c is any constant

4. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

5.$\int_a^b c\,f(x)dx = c\int_a^b f(x)dx$, where c is any constant

6. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

7. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx \ \ where \ \ a < b < c$

8. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof: Put $x=a-z$ Then $dx=-dz$

When $x=0,\ z=a$ and when $x=a,\ z=0$.

Therefore

$$\begin{aligned}\int_0^a f(x)\,dx &= -\int_a^0 f(a-z)\,dz\\ &= \int_0^a f(a-z)\,dz\\ &= \int_0^a f(a-x)\,dx\end{aligned}$$

9. $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \ \ if \ \ f(2a-x)=f(x)$

Proof: We know that

$$\begin{aligned}\int_0^{2a} f(x)\,dx &= \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx\\ &= \int_0^a f(x)\,dx - \int_a^0 f(2a-z)\,dz\\ &= \int_0^a f(x)\,dx + \int_0^a f(2a-z)\,dz\\ &= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx\\ &= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx \quad \text{Since } f(2a-x)=f(x)\end{aligned}$$

In the second integral of RHS Put $x=2a-z$. Then $dx=-dz$

When $x=a,\ z=a$ and when $x=2a,\ z=0$.

$$= 2\int_{0}^{a} f(x)\, dx$$

COMPARISON PROPERTIES OF THE INTEGRAL

10. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f\,(x)dx \geq 0$.

11. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f\,(x)dx \geq \int_a^b g\,(x)dx$.

12. If $m \leq f(x) \leq M \quad for \quad a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f\,(x)dx \leq M(b-a)$$

Example :

Use the properties of integrals to evaluate $\int_0^1 (\,4 + 3x^2)dx$.

Using Properties 4 and 5 of integrals, we have

$$\int_0^1 (\,4 + 3x^2)dx = \int_0^1 4\,dx + \int_0^1 3\,x^2 dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

$$= 4 + 3 \cdot \frac{1}{3} = 5$$

Example: If $\int_0^{10} f\,(x)dx = 17$ and $\int_0^8 f\,(x)dx = 12$, find $\int_8^{10} f\,(x)dx$.

We have

$$\int_0^8 f\,(x)dx + \int_8^{10} f\,(x)dx = \int_0^{10} f\,(x)dx$$

so $\int_8^{10} f\,(x)dx = \int_0^{10} f\,(x)dx - \int_0^8 f\,(x)dx = 17 - 12 = 5$

THE FUNDAMENTAL THEOREM OF CALCULUS- I

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f\,(t)dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) , and $g^{'}(x) = f(x)$.

Example:Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

Sol: Since $f(t) = \sqrt{1+t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g^{'}(x) = \sqrt{1+x^2}$$

Example: Find $\frac{d}{dx}\int_1^{x^4} \sec\,tdt$.

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Let $u = x^4$. Then

$$\frac{d}{dx}\int_1^{x^4} \sec\ tdt = \frac{d}{dx}\int_1^{u} \sec\ tdt$$
$$= \frac{d}{du}\left[\int_1^{u} \sec\ tdt\right]\frac{du}{dx}$$
$$= \sec u\frac{du}{dx}$$
$$= \sec(x^4)\cdot 4x^3$$

THE FUNDAMENTAL THEOREM OF CALCULUS-2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F^{'} = f$.

Example: Evaluate the integral $\int_1^3 e^x\, dx$.

The function $f(x) = e^x$ is continuous everywhere and we know that an anti- derivative is $F(x) = e^x$, so Part 2 of the Fundamental Theorem gives

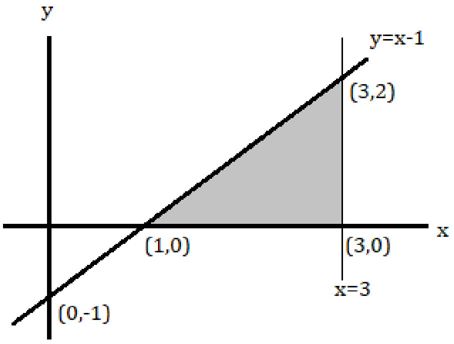
$$\int_1^3 e^x\, dx = F(3) - F(1) = e^3 - e$$

Example: Find the area under the parabola $y = x^2$ from 0 to 1.

An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$. The required area A is found using Part 2 of the Fundamental Theorem:

$$A = \int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Evaluate $\int_0^3 x - 1dx$ by interpreting in terms of the area.



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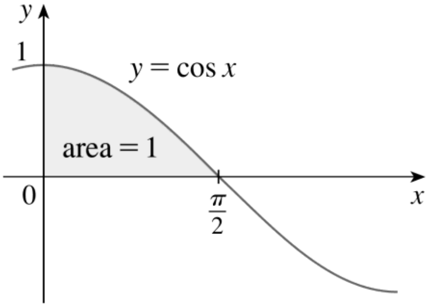
$$\int_0^3 x-1\,dx=\left[\frac{x^2}{2}-x\right]_0^3$$
$$=\frac{9}{2}-3$$
$$=\frac{3}{2}$$

This is the area bounded by the line $y=x-1, x-$axis and the line $x=3$.

Example: Find the area under the cosine curve from 0 to b, where $0 \le b \le \pi/2$.
Since an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$, we have

$$A=\int_0^b \cos\ x dx = [\sin x]_0^b$$
$$= \sin b — \sin 0$$
$$= \sin b$$



In particular, taking $b = \pi/2$, we have proved that the area under the cosine curve from 0 to $\pi/2$ is $\sin(\pi/2) = 1$.

Example : What is wrong with the following calculation?

$$\int_{-1}^{3}\frac{1}{x^2}dx=\left[\frac{x^{-1}}{-1}\right]_{-1}^{3}=-\frac{1}{3}-1=-\frac{4}{3}$$

we notice that this calculation must be wrong because the answer is negative but $f(x) = 1/x^2 \ge 0$ and Property 6 of integrals says that $\int_a^b f(x)dx \ge 0$ when $f \ge 0$. The Fundamental Theorem of Calculus applies to continuous functions. It can't be applied here because $f(x) = 1/x^2$ is not continuous on [−1, 3]. In fact, f has an infinite discontinuity at $x = 0$, so$\int_{-1}^{3}\frac{1}{x^2}dx$ does not exist

Solved Problems for Indefinite Integrals

1. **Find the general indefinite integral $\int(10x^4 - 2\sec^2 x)dx$**

$$\int(10x^4-2\sec^2 x)dx = 10\int x^4\,dx - 2\int \sec^2 x dx$$
$$=10\frac{x^5}{5}-2\tan x + C$$
$$=2x^5 - 2\tan x + C$$

2. **Evaluate$\int \frac{\cos\theta}{\sin^2\theta}d\theta$.**

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$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \quad \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}\,d\theta$$

$$= \int \operatorname{cosec}\theta\,\cot\theta d\theta = -\operatorname{cosec}\theta + C$$

3. $\int \frac{x+\sin x}{1+\cos x}dx$

$$\int \frac{x+\sin x}{1+\cos x}dx = \int \frac{x+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx$$

sin2x=2sinxcosx, cos²x = ½ (1+cos2x)

$$=\frac{1}{2}\int x\sec^2\frac{x}{2}\,dx + \int \tan\frac{x}{2}dx$$

In the first integral

$$Let\ u = x, \qquad dv = \frac{1}{2}\sec^2\frac{x}{2}\,dx$$

$$du = dx, \quad v = \tan\frac{x}{2}$$

$u\sin g\ \text{int}egration\ by\ parts,$

$$\int \frac{x+\sin x}{1+\cos x}dx = x.\tan\frac{x}{2} - \int \tan\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$$

$$= x.\tan\frac{x}{2}$$

THE SUBSTITUTION RULE

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int (g(x))g'(x)\,dx = \int (u)\,du$$

Example: Find $\int x^3 \cos(x^4+2)dx$.

We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using $x^3dx = du/4$ and the Substitution Rule, we have

$$\int x^3\ \cos(x^4+2)dx = \int \cos u \cdot \frac{1}{4}du = \frac{1}{4}\int \cos udu$$

$$=\tfrac{1}{4}\ \sin u + C$$

$$=\tfrac{1}{4}\ \sin(x^4+2) + C$$

Example: Evaluate $\int \sqrt{2x+1}\,dx$.

Let $u = 2x + 1$. Then $du = 2dx$, so $dx = du/2$. Thus the Substitution Rule gives

$$\int \sqrt{2x+1}\,dx. = \int \sqrt{u}\,dx.\frac{du}{2}$$

$$= \frac{1}{2} . \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{1}{3} u^{3/2} + C$$

$$= \frac{1}{3} (2x + 1)^{3/2} + C$$

Example: Find $\int \frac{x}{\sqrt{1-4x^2}} dx$.

Let $u = 1 - 4x^2$. Then $du = -8xdx$, so $xdx = -8du$ and

$$\int \frac{x}{\sqrt{1 - 4x^2}} dx = -\frac{1}{8} \int \frac{1}{\sqrt{u}} du$$

$$= -\frac{1}{8} \int u^{-1/2} du$$

$$= -\frac{1}{8} (2\sqrt{u}) + C$$

$$= -\frac{1}{4} \sqrt{1 - 4x^2} + C$$

Example Find $\int \sqrt{1 + x^2}\, x^5 dx$.

An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$.

Let $u = 1 + x^2$. Then $du = 2xdx$, so $xdx = du/2$.

Also $x^2 = u - 1$, so $x^4 = (u - 1)^2$:

$$\int \sqrt{1 + x^2}\, x^5 dx = \int \sqrt{1 + x^2}\, x^4 \cdot xdx$$

$$= \int \sqrt{u}\, (u - 1)^2 \frac{du}{2}$$

$$= \frac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2}) du$$

$$= \frac{1}{2} (\frac{2}{7} u^{7/2} - 2 . \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2}) + C$$

$$= \frac{1}{7} (1 + x^2)^{7/2} - \frac{2}{5} (1 + x^2)^{5/2} + \frac{1}{3} (1 + x^2)^{3/2} + C$$

Evaluate $I = \int \frac{x}{\left(x^2 - 1\right)\sqrt{x^2 + 1}} dx$

Evaluate $I = \int \frac{1}{\left(1 - x^2\right)\sqrt{x^2 + 1}} dx$

Let

$$\sqrt{x^2+1}=z$$
$$x^2+1=z^2$$
$$x^2=z^2-1$$
$$2x\ dx=2z\ dz$$
$$I=\int\frac{x}{(x^2-1)\sqrt{x^2+1}}\,dx$$
$$=\int\frac{z}{(z^2-2).z}\,dz$$
$$=\int\frac{1}{z^2-(\sqrt{2})^2}\,dz$$
$$=\frac{1}{2\sqrt{2}}\log\frac{z-\sqrt{2}}{z+\sqrt{2}}$$
$$=\frac{1}{2\sqrt{2}}\log\frac{\sqrt{x^2+1}-\sqrt{2}}{\sqrt{x^2+1}+\sqrt{2}}$$

$$Put\ x=\frac{1}{y},\ then\ \ dx=-\frac{1}{y^2}dy$$
$$I=-\int\frac{1}{y^2\left(1-\frac{1}{y^2}\right)\sqrt{1+\frac{1}{y^2}}}\,dy$$
$$=-\int\frac{y}{(y^2-1)\sqrt{y^2+1}}$$

By the previous example, we have

$$=-\frac{1}{2\sqrt{2}}\log\frac{\sqrt{y^2+1}-\sqrt{2}}{\sqrt{y^2+1}+\sqrt{2}}$$
$$=-\frac{1}{2\sqrt{2}}\log\frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{2}}{\sqrt{\frac{1}{x^2}+1}+\sqrt{2}}$$
$$=-\frac{1}{2\sqrt{2}}\log\frac{\sqrt{x^2+1}-x\sqrt{2}}{\sqrt{x^2+1}+x\sqrt{2}}$$

THE SUBSTITUTION RULE FOR DEFINITE INTEGRALS

If g' is continuous on$[a,b]$ and f is continuous on the range of $u=g(x)$

$$\int_a^b f\,(g(x))g'(x)dx=\int_{g(a)}^{g(b)} f\,(u)du$$

Example: Evaluate $\int_0^4\sqrt{2x+1}\,dx$.

We have $u=2x+1$ and $dx=du/2$. To find the new limits of integration.
when $x=0,\ \ u=2(0)+1=1$ and when $x=4,\ \ u=2(4)+1=9$

Therefore $\int_0^4\sqrt{2x+1}\,dx=\int_1^9\frac{1}{2}\sqrt{u}du$

$$=\frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^9$$
$$=\frac{1}{3}\left(9^{\frac{3}{2}}-1^{\frac{3}{2}}\right)$$
$$=\frac{26}{3}$$

Example: Evaluate $\int_1^2\frac{dx}{(3-5x)^2}$.

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Let $u = 3 - 5x$. Then $du = -5dx$, so $dx = -du/5$.
When $x = 1$, $u = -2$ and when $= 2$, $u = -7$. Thus

$$\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2}$$

$$= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-7}^{-2}$$

$$= \frac{1}{5}(-\frac{1}{7} + \frac{1}{2}) = \frac{1}{14}$$

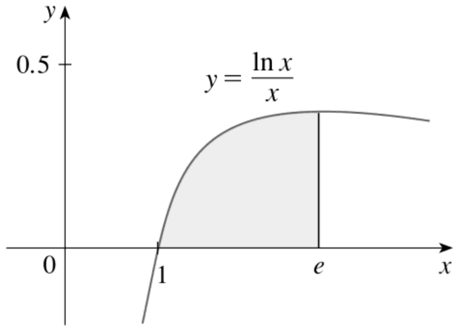
Example: Calculate $\int_1^e \frac{\ln x}{x} dx$.

let $u = \ln x$. Then $du = dx/x$.
When $x = 1, u = \ln 1 = 0$; when $x = e, u = \ln e = 1$.
Thus

$$\int_1^e \frac{\ln x}{x} dx = \int_0^1 u\, du$$

$$= \left[\frac{u^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$



RATIONALIZING SUBSTITUTIONS

Some non rational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective. Other instances appear in the exercises.

Example: Evaluate $\int \frac{\sqrt{x+4}}{x} dx$.

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2udu$.
Therefore

$$\int \frac{\sqrt{x+4}}{x} dx = \int \frac{u}{u^2-4} 2udu$$

$$= 2\int \frac{u^2}{u^2-4} du$$

$$= 2\int \left(1 + \frac{4}{u^2-4}\right) du$$

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$$= 2\int du + 8\int \frac{du}{u^2 - 4}$$

$$= 2u + 8 \cdot \frac{1}{2 \cdot 2} \ln\left|\frac{u-2}{u+2}\right| + C$$

$$= 2\sqrt{x+4} + 2 \text{ In}\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$$

INTEGRALS OF SYMMETRIC FUNCTIONS

Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)dx = 0$.

INTEGRATION BY PARTS: When integration by substitution is not possible, this method will be useful.

Formula for indefinite integrals	Formula for definite integrals
$\int u\,dv = uv - \int v\,du$	$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

.

Solved Problems

1. Evaluate by using integration by parts $\int xe^{4x}\,dx$

Let

$$u = x \quad and \quad dv = e^{4x}\,dx$$

$$du = dx \quad and \quad v = \frac{e^{4x}}{4}$$

$$\int xe^{4x}\,dx = x\frac{e^{4x}}{4} - \int \frac{e^{4x}}{4}dx$$

$$= x\frac{e^{4x}}{4} - \frac{1}{4}\int e^{4x}\,dx$$

$$= x\frac{e^{4x}}{4} - \frac{1}{4}\frac{e^{4x}}{4}$$

2. Evaluate by using integration by parts $\int_{-1}^{1} xe^{4x}\,dx$

Let

$$u = x \quad and \quad dv = e^{4x}\,dx$$

$$du = dx \quad and \quad v = \frac{e^{4x}}{4}$$

$$\int_{-1}^{1} xe^{4x}\,dx = \left[x\frac{e^{4x}}{4}\right]_{-1}^{1} - \int_{-1}^{1} \frac{e^{4x}}{4}dx$$

$$= \left[\frac{e^{4}}{4} + \frac{e^{-4}}{4}\right] - \frac{1}{4}\left[\frac{e^{4x}}{4}\right]_{-1}^{1}$$

$$=\left[\frac{e^4}{4}+\frac{e^{-4}}{4}\right]-\frac{1}{4}\left[\frac{e^4}{4}-\frac{e^{-4}}{4}\right]$$

3. Evaluate $\int x\sqrt{x+1}\ dx$ using integration by parts.

Let

$$u=x \qquad and \qquad dv=\sqrt{x+1}\ dx$$

$$du=dx \quad and \quad v=\frac{2}{3}(x+1)^{\frac{3}{2}}$$

$$\int x\sqrt{x+1}\ dx=x.\frac{2}{3}(x+1)^{\frac{3}{2}}-\int \frac{2}{3}(x+1)^{\frac{3}{2}}dx$$

$$=x.\frac{2}{3}(x+1)^{\frac{3}{2}}-\frac{2}{3}\frac{(x+1)^{\frac{5}{2}}}{\frac{5}{2}}$$

$$=\frac{2}{3}x(x+1)^{\frac{3}{2}}-\frac{4}{15}(x+1)^{\frac{5}{2}}+c$$

4. Evaluate $\int x\sqrt{x+1}\ dx$ using substitution method.

Let $u=x+1,\ x=u-1\ \ and\ \ dx=du$

$$\int x\sqrt{x+1}\ dx=\int (u-1)\sqrt{u}\ du$$

$$=\int (u-1)u^{\frac{1}{2}}\ du$$

$$=\int u^{\frac{3}{2}}-u^{\frac{1}{2}}\ du$$

$$=\frac{u^{\frac{5}{2}}}{\frac{5}{2}}-\frac{u^{\frac{3}{2}}}{\frac{3}{2}}$$

$$=\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{2}{3}(x+1)^{\frac{3}{2}}+c$$

5. Find $\int x\ \sin xdx$.

By Integration by parts, we have

$$\int x.\sin x\ dx=uv-\int vdu$$

$$=x\left(-\cos x\right)-\int -\cos x\ dx$$

$$=-x\cos x+\sin x+c$$

Let

$$u=x \qquad and \quad dv=\sin x\ dx$$

$$du=dx\ \ and \quad v=-\cos x$$

6. Find $\int \boldsymbol{lnx\ dx}$.

By Integration by parts, we have

Let

$$\int \ln x\ dx = uv - \int v du$$

$$= x.\ln x - \int x.\frac{1}{x}\ dx$$

$$= x\ln x - x + c$$

$$u = \ln x \quad and \quad dv = dx$$

$$du = \frac{1}{x}dx \ \ and \quad v = x$$

.

7. Evaluate: $\int x\sin\frac{x}{2}\cos\frac{x}{2}\cos xdx$

$$I = \frac{1}{2}\int x\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)\cos xdx$$

$$= \frac{1}{2}\int x\sin x\cos x\ dx$$

$$= \frac{1}{2\times 2}\int x\left(2\sin x\cos x\right)\ dx$$

$$= \frac{1}{4}\int x.\sin 2x\ dx$$

$$= \frac{1}{4}\left[-\frac{1}{2}x.\cos 2x - \int -\frac{1}{2}\cos 2x\ dx\right]$$

$$= \frac{1}{4}\left[-\frac{1}{2}x.\cos 2x + \frac{1}{2\times 2}\sin 2x\right]$$

$$= -\frac{1}{8}x\cos 2x + \frac{1}{16}\sin 2x$$

Let

$$u = x \quad and \quad dv = \sin 2x\ dx$$

$$du = dx \quad and \quad v = -\frac{1}{2}\cos 2x$$

$$\sin 2x = 2\sin x\cos x$$

.

8. Evaluate $\int\ x^5\sqrt{x^3+1}\ dx$ **using integration by parts**.

Let

$$u = x^3 \quad and \quad dv = x^2\sqrt{x^3+1}\ dx$$

$$du = 3x^2dx \quad and \quad dv = \frac{1}{3}\sqrt{x^3+1}\ d\left(x^3\right)$$

$$v = \frac{2}{3\times 3}\left(x^3+1\right)^{\frac{3}{2}}$$

$$\int\ x^5\sqrt{x^3+1}\ dx = x^3.\frac{2}{9}\left(x^3+1\right)^{\frac{3}{2}} - \int\ \frac{2}{9}\left(x^3+1\right)^{\frac{3}{2}}.3x^2\ dx$$

9. Evaluate $\int\ e^x\cos x\ dx$ **using integration by parts.**

Let

$$u = \cos x \quad and \quad dv = e^x dx$$

$$du = -\sin xdx \quad and \quad v = e^x$$

$$\int\ e^x\cos x\ dx = e^x\cos x + \int e^x\sin x\,dx$$

Let

$$u = \sin x \quad and \quad dv = e^x dx$$

$$du = \cos xdx \quad and \quad v = e^x$$

$$=\frac{2}{9}x^3\left(x^3+1\right)^{\frac{3}{2}}-\frac{6}{9\times 3}\int \left(x^3+1\right)^{\frac{3}{2}}\,d\left(x^3\right)$$

$$=\frac{2}{9}x^3\left(x^3+1\right)^{\frac{3}{2}}-\frac{6}{9\times 3}.\frac{2}{5}\left(x^3+1\right)^{\frac{5}{2}}$$

$$=\frac{2}{9}x^3\left(x^3+1\right)^{\frac{3}{2}}-\frac{4}{45}\left(x^3+1\right)^{\frac{5}{2}}+c$$

$$\int e^x\cos x\,dx=e^x\cos x+$$

$$e^x\sin x-\int e^x\cos x\,dx$$

$$2\int e^x\cos x\,dx=e^x\cos x+e^x\sin x$$

$$\int e^x\cos x\,dx=\frac{1}{2}e^x(\cos x+\sin x)+c$$

10. Evaluate: $\int x\sin^3 x\,dx$

$$I=\int x\sin^3 x\,dx$$

$$=\int x\left[\frac{1}{4}\left(3\sin x-\sin 3x\right)\right]dx$$

$$=\frac{3}{4}\int x\sin x\,dx-\frac{1}{4}\int x\sin 3x\,dx$$

$$=\frac{3}{4}\left[x\left(-\cos x\right)-\int -\cos x\,dx\right]-\frac{1}{4}\left[x\left(-\frac{1}{3}\cos 3x\right)-\int -\frac{1}{3}\cos 3x\,dx\right]$$

$$=\frac{3}{4}\left[-x\cos x+\sin x\right]-\frac{1}{4}\left[-\frac{x}{3}\cos 3x+\frac{1}{9}\sin 3x\right]$$

$$=\frac{1}{4}\left[-3x\cos x+3\sin x+\frac{1}{3}x\cos 3x-\frac{1}{9}\sin 3x\right]$$

Let

$u=x \quad and \quad dv=\sin x\,dx$

$du=dx \ \ and \quad v=-\cos x$

Let

$u=x \quad \& \quad dv=\sin 3x\,dx$

$du=dx \quad \& \quad v=-\frac{1}{3}\cos 3x$

11. Evaluate: $\int x\tan^2 x\,dx$

$$I=\int x\left(\sec^2 x-1\right)dx$$

$$=\int x\sec^2 x\,dx-\int x\,dx$$

$$=\left[x\tan x-\int \tan x\,dx\right]-\frac{x^2}{2}$$

$$=\left[x\tan x-(-\log\cos x)\right]-\frac{x^2}{2}$$

$$=x\tan x+\log\cos x-\frac{x^2}{2}$$

Let

$u=x \quad and \quad dv=\sec^2 x\,dx$

$du=dx \ \ and \quad v=\tan x$

$1+\tan^2 x=\sec^2 x$

12. Evaluate: $\int x\cot^2 x\ dx$

$$I=\int x\left(co\sec^2 x-1\right)\,dx$$

$$=\int xco\sec^2 x\ dx-\int x\ dx$$

$$=\left[-x\cot x-\int -\cot x\ dx\right]-\frac{x^2}{2}$$

$$=\left[-x\cot x+(\log\sin x)\right]-\frac{x^2}{2}$$

$$=-x\cot x+\log\sin x-\frac{x^2}{2}$$

Let

$u=x \quad and \quad dv=\cos ec^2x\ dx$

$du=dx \ \ and \quad v=-\cot x$

$$1+\cot^2 x=co\sec^2 x$$

13 Evaluate: $\int x\cot x\cos ec^2x\ dx$

$$I=\int x\left(\cot x\cos ec^2x\right)\,dx$$

$\therefore$ By integration by parts,

$$I=-\frac{1}{2}x\cot^2 x-\int -\frac{1}{2}\cot^2 x\ dx$$

$$=-\frac{1}{2}x\cot^2 x+\frac{1}{2}\int \cos ec^2x-1\ dx$$

$$=-\frac{1}{2}x\cot^2 x+\frac{1}{2}\left(-\cot x-x\right)$$

$$\boxed{\begin{array}{l}1+\cot^2 x=\cos ec^2x\\ D(\cot x)=-c\,os\,ec^2x\end{array}}$$

Let

$u=x \quad and \quad du=dx$

$dv=\cot x\cos ec^2x\ dx$

$v=\int \cot x\cos ec^2x\,dx$

If $\ y=\cot x,\ dy=-\cos ec^2x\,dx$

$$v=\int \cot x\cos ec^2x\,dx=\int -y\,dy$$

$$=-\frac{y^2}{2}$$

$$=-\frac{1}{2}\cot^2 x$$

14. Evaluate: $\int \log\left(1+x^2\right)\,dx$

Let

$u=\log\left(1+x^2\right) \qquad and \quad dv=dx$

$du=\dfrac{1}{\left(1+x^2\right)}.2x\ dx \ \ and \quad v=x$

$$I=\int \log\left(1+x^2\right)\,dx$$

$$=\log\left(1+x^2\right).x-\int x\frac{1}{1+x^2}2x\,dx$$

$$=x.\log\left(1+x^2\right)-2\int\frac{x^2}{1+x^2}\,dx$$

$$=x.\log\left(1+x^2\right)-2\int 1-\frac{1}{1+x^2}\,dx$$

$$=x.\log\left(1+x^2\right)-2\left[x-\tan^{-1}x\right]$$

.

15. Evaluate: $\int\left(\log x\right)^2\,dx$

$$I=\int\left(\log x\right)^2\,dx$$

$$=\left(\log x\right)^2.x-\int x.2\log x.\frac{1}{x}\,dx\quad (IBP)$$

$$=x.\log\left(\log x\right)^2-2\int\log x\,dx$$

$$=x.\log\left(\log x\right)^2-2\left[\log x.x-\int\frac{1}{x}.x\,dx\right]\quad (IBP)$$

$$=x.\log\left(\log x\right)^2-2\left[x.\log x-x\right]$$

Let

$u=\left(\log x\right)^2 \quad and \quad dv=dx$

$du=2.\log x.\frac{1}{x}\,dx \;\; and \quad v=x$

$u=\left(\log x\right) \quad and \quad dv=dx$

$du=\frac{1}{x}\,dx \quad and \quad v=x$

.

16. Evaluate: $\int x^n.\log x\,dx$

$$I=\int x^n\left(\log x\right)dx$$

$$=\left(\log x\right).\frac{x^{n+1}}{n+1}-\int\frac{1}{x}.\frac{x^{n+1}}{n+1}dx$$

$$=\left(\log x\right).\frac{x^{n+1}}{n+1}-\frac{1}{n+1}\int x^n dx$$

$$=\left(\log x\right).\frac{x^{n+1}}{n+1}-\frac{1}{n+1}.\frac{x^{n+1}}{n+1}$$

Let

$u=\log x \quad and \quad dv=x^n dx$

$du=\frac{1}{x}\,dx \quad and \quad v=\frac{x^{n+1}}{n+1}$

.

17 Evaluate: $\int\sin^{-1}x\,dx$

Let

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$$I=\int \sin^{-1}x\ dx$$

$$=\sin^{-1}x.x-\int x.\frac{1}{\sqrt{1-x^2}}\ dx$$

$$=x.\sin^{-1}x-\left(-\frac{1}{2}\right)\int\frac{1}{\sqrt{t}}du$$

$$=x.\sin^{-1}x+\frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}}$$

$$=x.\sin^{-1}x+\sqrt{1-x^2}$$

$$u=\sin^{-1}x \qquad and \quad dv=dx$$

$$du=\frac{1}{\sqrt{1-x^2}}\ dx \ \ and \quad v=x$$

Let

$$t=1-x^2$$

$$dt=-2x\ dx$$

$$\int\frac{1}{\sqrt{t}}dt=\int t^{-\frac{1}{2}}dt=\frac{t^{\frac{1}{2}}}{\frac{1}{2}}=2\sqrt{t}$$

.

18 Evaluate: $\int x\tan^{-1}x\ dx$

$$I=\int x\tan^{-1}x\ dx$$

$$=\tan^{-1}.\frac{x^2}{2}-\int\frac{x^2}{2}.\frac{1}{1+x^2}\ dx$$

$$=\frac{x^2}{2}.\tan^{-1}x-\frac{1}{2}\int 1-\frac{1}{1+x^2}\ dx$$

$$=\frac{x^2}{2}.\tan^{-1}x-\frac{1}{2}\left[x-\tan^{-1}x\right]$$

Let

$$u=ta\,\mathrm{n}^{-1}x \qquad and \quad dv=xdx$$

$$du=\frac{1}{1+x^2}\ dx \quad and \quad v=\frac{x^2}{2}$$

.

19 Evaluate: $\int x^2\tan^{-1}\frac{x}{2}\ dx$

$$I=\int x^2\tan^{-1}\frac{x}{2}\ dx$$

$$=\tan^{-1}\frac{x}{2}.\frac{x^3}{3}-\int\frac{x^3}{3}.\frac{2}{4+x^2}\ dx$$

$$=\tan^{-1}\frac{x}{2}.\frac{x^3}{3}-\frac{2}{3}\int\frac{x^3}{4+x^2}\ dx$$

$$=\frac{x^3}{3}.\tan^{-1}\frac{x}{2}-\frac{2}{3}\int x-\frac{4x}{4+x^2}\ dx$$

$$=\frac{x^3}{3}.\tan^{-1}\frac{x}{2}-\frac{2}{3}\int xdx+\frac{4}{3}\int\frac{2x}{4+x^2}\ dx$$

$$=\frac{x^3}{3}.\tan^{-1}\frac{x}{2}-\frac{2}{3}\frac{x^2}{2}+\frac{4}{3}\log\left(4+x^2\right)$$

Let

$$u=ta\,\mathrm{n}^{-1}\frac{x}{2} \qquad and \quad dv=x^2dx$$

$$du=\frac{1}{1+\frac{x^2}{4}}\frac{1}{2}\ dx \quad and \quad v=\frac{x^3}{3}$$

$$du=\frac{2}{4+x^2}\ dx$$

20. Evaluate $\int \cos^n x \, dx$ by using integration by parts.

$$\int \cos^n x \, dx = \int \cos^{n-1} x \ \cos x \, dx$$

$$= \cos^{n-1} x.\sin x + \int \sin x.(n-1)\cos^{n-2} x.\sin x \, dx$$

$$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x.\sin^2 x \, dx$$

$$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x.\left(1-\cos^2 x\right) dx$$

Let $u = \cos^{n-1} x$ and $dv = \cos x \, dx$

$$du = (n-1)\cos^{n-2} x(-\sin x)dx$$

$$= -(n-1)\cos^{n-2} x.\sin x \, dx$$

$$v = \sin x$$

$$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n x \, dx$$

$$= \cos^{n-1} x.\sin x + (n-1)I_{n-2} - (n-1)I_n \qquad \text{where } I_n = \int \cos^n x \, dx$$

$$I_n + (n-1)I_n = \cos^{n-1} x.\sin x + (n-1)I_{n-2}$$

$$nI_n = \cos^{n-1} x.\sin x + (n-1)I_{n-2}$$

$$I_n = \frac{1}{n}\left(\cos^{n-1} x.\sin x\right) + \frac{(n-1)}{n} I_{n-2}$$

21. Evaluate $\int \sin^n x \, dx$ by using integration by parts.

$$\int \sin^n x \, dx = \int \sin^{n-1} x \ \sin x \, dx$$

$$= -\sin^{n-1} x.\cos x - \int -\cos x.(n-1)\sin^{n-2} x.\cos x \, dx$$

$$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x.\cos^2 x \, dx$$

$$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x.\left(1-\sin^2 x\right) dx$$

Let $u = \sin^{n-1} x$ and $dv = \sin x \, dx$

$$du = (n-1)\sin^{n-2} x(\cos x)dx$$

$$= (n-1)\sin^{n-2} x.\cos x \, dx$$

$$v = -\cos x$$

$$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x \, dx \ - (n-1)\int \sin^n x \, dx$$

$$I_n = -\sin^{n-1} x.\cos x + (n-1)\ I_{n-2} - (n-1)\ I_n \quad \text{where } I_n = \int \sin^n x \, dx$$

$$I_n + (n-1)\ I_n = -\sin^{n-1} x.\cos x + (n-1)\ I_{n-2}$$

$$I_n(1+n-1) = -\sin^{n-1} x.\cos x + (n-1)\ I_{n-2}$$

$$nI_n = -\sin^{n-1} x.\cos x + (n-1)\ I_{n-2}$$

$$I_n = -\frac{1}{n}\sin^{n-1} x.\cos x + \frac{(n-1)}{n}\ I_{n-2}$$

22. Evaluate $\int x\sin^n x\, dx$ by using integration by parts.

Let $u = x\sin^{n-1} x$ and $dv = \sin x\, dx$

$du = \sin^{n-1} x + (n-1)x\sin^{n-2} x(\cos x)dx$ and $v = -\cos x$

$$\int x\sin^n x\, dx = \int x\sin^{n-1}x \ \sin x\, dx$$

$$= -x\sin^{n-1} x.\cos x - \int -\cos x.\left[\sin^{n-1} x + (n-1)x.\sin^{n-2} x.\cos x\right]dx$$

$$= -x\sin^{n-1} x.\cos x + \int \cos x.\sin^{n-1} x\, dx + (n-1)\int x\sin^{n-2} x.\cos^2 x\, dx$$

$$= -x\sin^{n-1} x.\cos x + \int \ \sin^{n-1} x\, d(\sin x) + (n-1)\int x\sin^{n-2} x.\left(1-\sin^2 x\right)dx$$

$$= -x\sin^{n-1} x.\cos x + \frac{\sin^n x}{n} + (n-1)\int x\sin^{n-2} x\, dx - (n-1)\int x\sin^n x\, dx$$

$$I_n = -x\sin^{n-1} x.\cos x + \frac{\sin^n x}{n} + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n\left(1+n-1\right) = -x\sin^{n-1} x.\cos x + \frac{\sin^n x}{n} + (n-1)I_{n-2}$$

$$I_n = -\frac{1}{n}x\sin^{n-1} x.\cos x + \frac{\sin^n x}{n^2} + \frac{(n-1)}{n}I_{n-2}$$

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

1. Evaluate $\int \cos^3 x dx$.

$$\int \cos^3 x dx = \int \cos^2 x \cdot \ \cos x dx$$

$$= \int (\, 1 - \sin^2 x) \cos x dx$$

$$= \int (\, 1 - u^2) du$$

$$= u - \frac{1}{3}u^3 + C$$

$$= \ \sin x - \frac{1}{3}\sin^3 x + C$$

2. Evaluate $\int_0^\pi \sin^2 x dx$.

$$\int_0^\pi \sin^2 x dx = \frac{1}{2}\int_0^\pi (\, 1 - \ \cos\ 2x) dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\ \sin 2x\right]_0^\pi$$

$$= \frac{1}{2}\left(\pi - \frac{1}{2}\sin 2\pi\right) - \frac{1}{2}\left(0 - \frac{1}{2}\sin 0\right) = \ \frac{1}{2}\pi$$

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3. Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\, dx$

By reduction formula,

$$\int_0^{\frac{\pi}{2}} \sin^6 x\, dx = \frac{5.3.1}{6.4.2}\frac{\pi}{2} = \frac{5\pi}{32}$$

4. Evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x\, dx$

By reduction formula,

$$\int_0^{\frac{\pi}{2}} \sin^6 x\, dx = \frac{5.3.1}{6.4.2} = \frac{5}{16}$$

5. Evaluate $\int_0^{\frac{\pi}{4}} \log\left(1+\tan\theta\right)d\theta$

Let

$$I = \int_0^{\frac{\pi}{4}} \log\left(1+\tan\theta\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta, \ By\ \ property$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log 2\ d\theta - \int_0^{\frac{\pi}{4}} \log\left(1+\tan\theta\right)d\theta$$

$$I\ = \int_0^{\frac{\pi}{4}} \log 2\ d\theta - I$$

$$2I\ =\ \frac{\pi}{4}\log 2$$

$$I\ \ =\ \frac{\pi}{8}\log 2$$

6. Evaluate $\int_0^1 \left[\tan^{-1}x + \tan^{-1}(1-x)\right] dx$

Let

$$I = \int_0^1 \tan^{-1}x\ dx + \int_0^1 \tan^{-1}(1-x)\ \ dx$$

$$= \int_0^1 \tan^{-1}x\ dx + \int_0^1 \tan^{-1}\left[1-\left(1-x\right)\right]\ dx$$

$$= 2\int_0^1 \tan^{-1}x\ dx$$

Let

$$u = \tan^{-1}x, \quad dv = dx$$

$$du = \frac{1}{1+x^2}, \quad v = x$$

$$I = 2\left[x.\tan^{-1}x\right]_0^1 - 2\int_0^1 \frac{x}{1+x^2}\ dx$$

$$= 2.\frac{\pi}{4} - \left[\log\left(1+x^2\right)\right]_0^1$$

$$= \frac{\pi}{2} - \left[\log 2 - \log 1\right]$$

$$= \frac{\pi}{2} - \log 2$$

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STRATEGY FOR EVALUATING $\int \sin^m x\cos^n x dx$

(a) If the power of cosine is odd $(n = 2k + 1)$, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x\cos^{2k+1} x dx = \int \sin^m x(\cos^2 x)^k \cos x dx$$

$$= \int \sin^m x(1 - \sin^2 x)^k \cos x dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd $(m = 2k + 1)$, save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x\cos^n x dx = \int (\sin^2 x)^k \cos^n x \sin x dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x dx$$

Then substitute $u = \cos x$.

[Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) , \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity $\sin x \cos x = \frac{1}{2} \sin 2x$

We can use a similar strategy to evaluate integrals of the form $\int \tan^m x\sec^n x dx$.

(a) Since $(d/dx) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

(b) Or, since $(d/dx) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

1. Evaluate $\int \sin^5 x dx$

$$\int \sin^5 x dx = \int \left(\sin^2 x\right)^2 \sin x \, dx$$

$$= \int \left(1 - \cos^2 x\right)^2 \sin x \, dx$$

$$= \int \left(1 - \cos^2 x\right)^2 \sin x \, dx$$

[Put $u = \cos x, \; du = -\sin x dx$]

$$\int \sin^5 x dx = -\int \left(1 - u^2\right)^2 du$$

2. Evaluate $\int \frac{\sin^7 x}{\cos^4 x} dx$

$$\int \frac{\sin^7 x}{\cos^4 x} dx = \int \frac{\left(\sin^2 x\right)^3}{\cos^4 x} \sin x \, dx$$

$$= \int \frac{\left(1 - \cos^2 x\right)^3}{\cos^4 x} \sin x \, dx$$

[Put $u = \cos x, \; du = -\sin x dx$]

$$= \int -1 - u^4 + 2u^2 \; du$$

$$= \left[-u - \frac{u^5}{5} + \frac{2u^3}{3} \right]$$

$$= \left[-\cos x - \frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} + c \right]$$

$$\int \frac{\sin^7 x}{\cos^4 x} dx = -\int \frac{\left(1-u^2\right)^3}{u^4} du$$

$$= -\int \frac{1}{u^4}\left(1 - u^6 + 3u^4 - 3u^2\right) du$$

$$= \int \; -u^{-4} + u^2 - 3 + 3u^{-2} \; du$$

$$= \left[-\frac{u^{-3}}{-3} + \frac{u^3}{3} - 3u + 3\frac{u^{-1}}{-1} \right]$$

$$= \left[\frac{1}{3u^3} + \frac{u^3}{3} - 3u - 3\frac{1}{u} \right]$$

$$= \left[\frac{1}{3\cos^3 x} + \frac{\cos^3 x}{3} - 3\cos x - 3\frac{1}{\cos x} \right]$$

3. Evaluate $\int \tan^6 x\sec^4 x dx$.

We can then evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x dx$:

$$\int \tan^6 x\sec^4 x dx = \int \tan^6 x\sec^2 x\sec^2 x dx$$

$$= \int \tan^6 x(1 + \tan^2 x)\sec^2 x dx$$

$$= \int u^6 (1 + u^2) du$$

$$= \int (u^6 + u^8) du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C$$

4. Find $\int \tan^5 \theta\sec^7 \theta d\theta$.

$$\int \tan^5 \theta\sec^7 \theta d\theta$$

$$= \int \tan^4 \theta\sec^6 \theta \sec\theta \tan\theta d\theta$$

$$= \int (\sec^2\theta - 1)^2 \sec^6\theta \sec\theta \tan\theta d\theta$$

$$= \int (u^2 - 1)^2 u^6 du$$

$$= \int (u^{10} - 2u^8 + u^6) du$$

If we separate a $\sec\theta \tan\theta$ factor, we can convert the remaining power of tangent to an expression involving only secant using the identity $\tan^2\theta = \sec^2\theta - 1$.

We can then evaluate the integral by substituting $u = \sec\theta$, so $du = \sec\theta \tan\theta d\theta$:

$$=\frac{u^{11}}{11}-2\frac{u^9}{9}+\frac{u^7}{7}+C$$

$$=\frac{1}{11}\sec^{11}\theta-\frac{2}{9}\sec^9\theta+\frac{1}{7}\sec^7\theta+C$$

5 Find $\int \tan^3 xdx$.

Here only $\tan x$ occurs, so we use $\tan^2x = \sec^2x - 1$ to rewrite a $\tan^2x$ factor in terms of $\sec^2x$:

$$\int \tan^3 xdx = \int \tan\ x\tan^2xdx$$

$$= \int \tan\ x(\sec^2x - 1)dx$$

$$= \int \tan x\sec^2xdx - \int \tan\ xdx$$

$$= \int \tan xd(tanx) - \int \tan\ xdx$$

$$= \frac{\tan^2x}{2} - \ \ln|\sec x| + C$$

To evaluate the integrals

$(a)\int sinmx\cos nxdx$,

$(b)\int sinmx\sin nxdx$, or

(c) $\int \cos mx\cos nxdx$, use the corresponding identity:

$$\text{(a)}\ \sin A\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$\text{(b)}\ \sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\text{(c)}\ \cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

Example: Evaluate $\int \sin 4x\cos 5xdx$.

This integral could be evaluated using integration by parts, but it's easier to use the identity as follows:

$$\int \sin 4x\cos 5xdx = \int \frac{1}{2}[\sin(-x) + \sin 9x]dx$$

$$= \frac{1}{2}\int(-\sin x + \sin 9x)dx$$

$$= \frac{1}{2}(\cos x - \frac{1}{9}\cos 9x) + C$$

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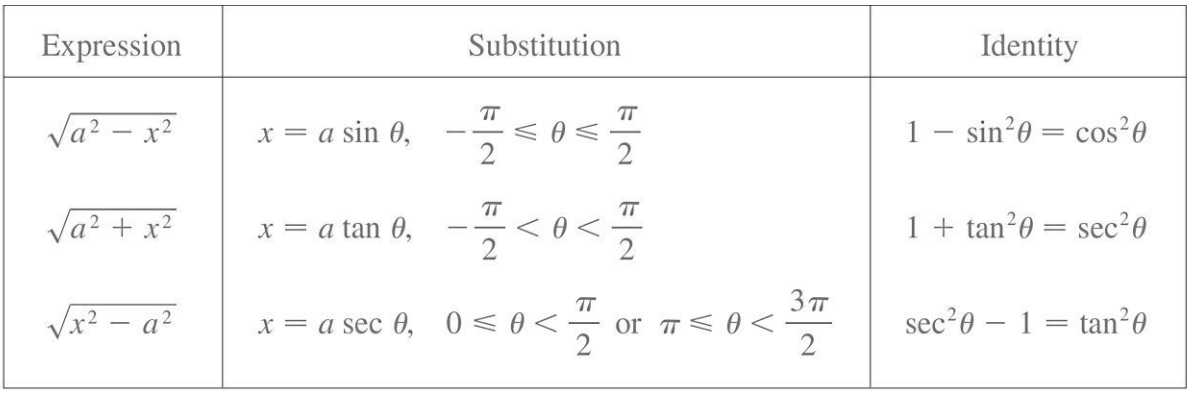
INTEGRATION BY TRIGONOMETRIC SUBSTITUTIONS



Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

1. **Evaluate** $\int \frac{\sqrt{9-x^2}}{x^2} dx$.

$$\int \frac{\sqrt{9-x^2}}{x^2} dx = \int \frac{3 \cos \theta}{9\sin^2\theta} 3 \cos \theta d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta} d\theta$$

$$= \int \cot^2 \theta d\theta$$

$$= \int (\operatorname{cosec}^2\theta - 1) d\theta$$

$$= -\cot \theta - \theta + C$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\frac{x}{3} + C$$

Let $x = 3 \sin \theta$, Then $dx = 3 \cos \theta d\theta$

$$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = 3 \cos \theta$$

Since $\sin \theta = \frac{x}{3}$, $\cos \theta = \sqrt{1 - sin^2\theta}$

$$= \sqrt{1 - \frac{x^2}{9}}$$

$$= \sqrt{\frac{9-x^2}{9}}$$

$$\cot \theta = \frac{cos\theta}{sin\theta} = \frac{\sqrt{9-x^2}}{3}\frac{3}{x} = \frac{\sqrt{9-x^2}}{x}$$

Since $\sin \theta = x/3$, we have $\theta = \sin^{-1}(x/3)$

.

2. **Evaluate** $\int \frac{1}{x^4\sqrt{9-x^2}} dx$

$$\int \frac{1}{x^4\sqrt{9-x^2}} dx = \int \frac{3\cos\theta}{3^4 \sin^4 \theta . 3\cos\theta} d\theta$$

$$= \int \frac{1}{3^4 \sin^4 \theta} d\theta$$

Let $x = 3 \sin \theta$, Then $dx = 3 \cos\theta \ d\theta$

$$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = 3 \cos \theta$$

WKT $1 + \cot^2 \theta = \cos ec^2\theta$

Put $\cot \theta = u; \quad \therefore du = -\cos ec^2\theta \, d\theta$

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$$=\frac{1}{81}\int \operatorname{cos}ec^{4}\theta\ d\theta$$

$$=\frac{1}{81}\int \operatorname{cos}ec^{2}\theta.\operatorname{cos}ec^{2}\theta\ d\theta$$

$$=\frac{1}{81}\int \left(1+\cot^{2}\theta\right).\operatorname{cos}ec^{2}\theta\ d\theta$$

$$=-\frac{1}{81}\int \left(1+u^{2}\right)du$$

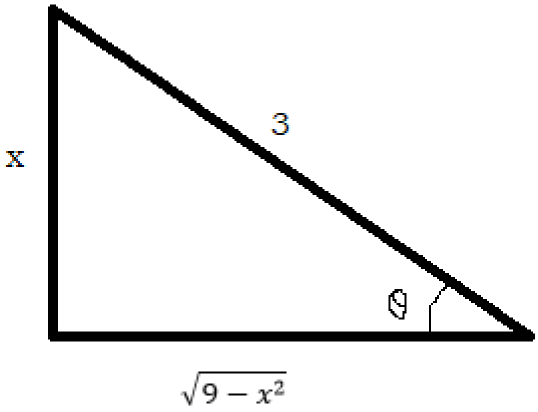
$$=-\frac{1}{81}\left[u+\frac{u^{3}}{3}\right]$$

$$=-\frac{1}{81}\left[\cot\theta+\frac{\cot^{3}\theta}{3}\right]$$

$$=-\frac{1}{81}\left[\frac{\sqrt{9-x^{2}}}{x}+\frac{1}{3}\frac{\left(9-x^{2}\right)^{\frac{3}{2}}}{x^{3}}\right]+c$$

From the assumption $\sin\theta=\frac{x}{3}$

$\therefore\ \tan\theta=\frac{x}{\sqrt{9-x^{2}}}$ and $\cot\theta=\frac{\sqrt{9-x^{2}}}{x}$



3 Find $\int\frac{1}{x^{2}\sqrt{x^{2}+4}}dx$.

$$\int\frac{dx}{x^{2}\sqrt{x^{2}+4}}=\int\frac{2\sec^{2}\theta d\theta}{4\tan^{2}\theta\cdot 2\sec\theta}$$

$$=\frac{1}{4}\int\frac{\sec\theta}{\tan^{2}\theta}d\theta$$

$$=\frac{1}{4}\int\frac{\cos\theta}{\sin^{2}\theta}d\theta$$

$$=\frac{1}{4}\int\frac{du}{u^{2}}$$

$$=\frac{1}{4}\left(-\frac{1}{u}\right)+C$$

$$=-\frac{1}{4\sin\theta}+C$$

$$=-\frac{\operatorname{cosec}\theta}{4}+C$$

$$=-\frac{\sqrt{x^{2}+4}}{4x}+C$$

Let $x=2\tan\theta$, Then $dx=2\sec^{2}\theta d\theta$

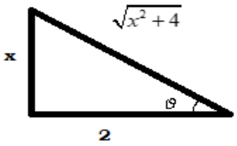
$$\sqrt{x^{2}+4}=\sqrt{4(\tan^{2}\theta+1)}$$
$$=\sqrt{4\sec^{2}\theta}$$
$$=2\sec\theta$$

$$\frac{\sec\theta}{\tan^{2}\theta}=\frac{1}{\cos\theta}\cdot\frac{\cos^{2}\theta}{\sin^{2}\theta}=\frac{\cos\theta}{\sin^{2}\theta}$$

Let $u=\sin\theta,\ du=cos\theta d\theta$

Since $\tan\theta=\frac{x}{2}$, $\sin\theta=\frac{x}{\sqrt{x^{2}+4}}$

$$\therefore\ \operatorname{cosec}\theta=\frac{\sqrt{x^{2}+4}}{x}$$



4 Evaluate $\int \frac{\sqrt{25x^2-4}}{x}dx$

$$\int \frac{\sqrt{25x^2-4}}{x}dx = 5\int \frac{\sqrt{x^2-\frac{4}{25}}}{x}\ dx$$

$$= 5\int \frac{\frac{2}{5}\tan\theta}{\frac{2}{5}\sec\theta}\ \frac{2}{5}\sec\theta\tan\theta\ d\theta$$

$$= 5.\frac{2}{5}\int \tan^2\theta\ d\theta$$

$$= 5.\frac{2}{5}\int \tan^2\theta\ d\theta$$

$$= 2\int \sec^2\theta - 1\ \ d\theta$$

$$= 2\ \left[\tan\theta - \theta\right]$$

$$= 2\left[\frac{\sqrt{25x^2-4}}{2} - \sec^{-1}\frac{5x}{2}\right]$$

Put $x = \frac{2}{5}\sec\theta \qquad \therefore\ dx = \frac{2}{5}\sec\theta\tan\theta\ d\theta$

$$x^2 - \frac{4}{25} = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$

$$= \frac{4}{25}\left(\sec^2\theta - 1\right)$$

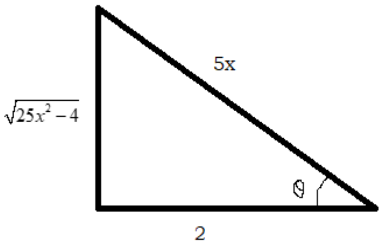
$$= \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2-\frac{4}{25}} = \sqrt{\frac{4}{25}\tan^2\theta} = \frac{2}{5}\tan\theta$$

WKT $1+\tan^2\theta = \sec^2\theta$

From our assumption $\frac{5x}{2} = \sec\theta$

$$\frac{2}{5x} = \cos\theta$$



$\theta = \sec^{-1}\frac{5x}{2}$ and $\tan\theta = \frac{\sqrt{25x^2-4}}{2}$

----

5 Evaluate $\int_{-\frac{4}{5}}^{-\frac{2}{5}} \frac{\sqrt{25x^2-4}}{x}\ dx$

From the previous example, we have

when $x = -\frac{4}{5},\ -\frac{4}{5} = \frac{2}{5}\sec\theta$

$$-\frac{4}{5}\frac{5}{2} = \frac{1}{\cos\theta}$$

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$$\int_{-\frac{4}{5}}^{-\frac{2}{5}} \frac{\sqrt{25x^2-4}}{x}\, dx = -2 \int_{\frac{2\pi}{3}}^{\pi} \sec^2\theta - 1 \;\; d\theta$$

$$= -2 \;\left[\tan\theta - \theta\right]_{\frac{2\pi}{3}}^{\pi}$$

$$= -2 \left[\left(\tan\pi - \pi\right) - \left(\tan\frac{2\pi}{3} - \frac{2\pi}{3}\right)\right]$$

$$= -2 \left[\left(0 - \pi\right) - \left(-\sqrt{3} - \frac{2\pi}{3}\right)\right]$$

$$= 2\left(\frac{\pi}{3} - \sqrt{3}\right)$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

when $x = -\frac{2}{5},\;\; -\frac{2}{5} = \frac{2}{5}\sec\theta$

$$-\frac{2}{5}\frac{5}{2} = \frac{1}{\cos\theta}$$

$$\cos\theta = -1$$

$$\theta = \cos^{-1}\left(-1\right) = \pi$$

In the range of $\frac{2\pi}{3} < \theta < \pi,$ the tangent is negative. Therefore

$$\sqrt{x^2 - \frac{4}{25}} = \sqrt{\frac{4}{25}\tan^2\theta} = -\frac{2}{5}\tan\theta$$

.

6 Evaluate: $\int \frac{x^2 \tan^{-1} x}{1+x^2}\, dx$

Let $x = \tan\theta,$ *the* $dx = \sec^2\theta\; d\theta$

and $\theta = \tan^{-1} x.$

$1 + \tan^2\theta = \sec^2\theta$

IBP

Let

$u = \theta \qquad$ *and* $\quad dv = \sec^2\theta\; d\theta$

$du = d\theta \quad$ *and* $\qquad v = \tan\theta$

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$$I=\int \frac{x^2\tan^{-1}x}{1+x^2}\,dx$$

$$=\int\frac{\tan^2\theta\,.\,\theta}{1+\tan^2\theta}\sec^2\theta\,d\theta$$

$$=\int\frac{\tan^2\theta\,.\,\theta}{\sec^2\theta}\sec^2\theta\,d\theta$$

$$=\int\left(\sec^2\theta-1\right).\,\theta\,d\theta$$

$$=\int\theta.\sec^2\theta\,d\theta-\int\theta\,d\theta$$

$$=\left[\theta.\tan\theta\right]-\int\tan\theta\,d\theta-\frac{\theta^2}{2}\quad(IBP)$$

$$=\left[\theta.\tan\theta\right]-\log(\sec\theta)-\frac{\theta^2}{2}$$

$$=\left[\theta.\tan\theta\right]-\log\sqrt{1+\tan^2\theta}-\frac{\theta^2}{2}$$

$$=\left[\theta.\tan\theta\right]-\frac{1}{2}\log\left(1+\tan^2\theta\right)-\frac{\theta^2}{2}$$

$$=\left[x.\tan^{-1}x\right]-\frac{1}{2}\log\left(1+x^2\right)-\frac{1}{2}\left(\tan^{-1}x\right)^2$$

.

7. Evaluate: $\int_0^\infty \frac{x\tan^{-1}x}{\left(1+x^2\right)^2}\,dx$

$Let\ x=\tan\theta,\ the\ dx=\sec^2\theta\,d\theta$

$and\ \theta=\tan^{-1}x.$

$1+\tan^2\theta=\sec^2\theta$

IBP

Let

$u=\theta\quad and\quad dv=\sin 2\theta\,d\theta$

$du=d\theta\quad and\quad v=-\frac{\cos 2\theta}{2}$

$\cos\pi=-1,\quad \sin 0=\sin\pi=0$

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$$I = \int_0^\infty \frac{x\tan^{-1}x}{\left(1+x^2\right)^2}\,dx$$

$$= \int_0^{\pi/2} \frac{\tan\theta\,.\,\theta}{\left(1+\tan^2\theta\right)^2}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{\tan\theta\,.\,\theta}{\sec^4\theta}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{\tan\theta\,.\,\theta}{\sec^2\theta}\,d\theta$$

$$= \int_0^{\pi/2} \theta.\frac{\sin\theta}{\cos\theta}.\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} \theta.\sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \theta.\sin 2\theta\,d\theta$$

$$= \frac{1}{2}\left[\theta.\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} \frac{-\cos 2\theta}{2}\,d\theta \quad (IBP)$$

$$= -\frac{1}{4}\left[\theta.\cos 2\theta\right]_0^{\pi/2} + \frac{1}{4}\int_0^{\pi/2} \cos 2\theta\,d\theta$$

$$= -\frac{1}{4}\left[\frac{\pi}{2}\cos\pi\right] + \frac{1}{4}\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= -\frac{1}{4}\left[-\frac{\pi}{2}\right]$$

$$= \frac{\pi}{8}$$

.

8 **Evaluate:** $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

9 **Evaluate:** $\int \frac{\sin^{-1}x}{\left(1-x^2\right)^{3/2}}\,dx$

$$I=\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$$

$$Put\ x=\sin\theta,\ dx=\cos\theta\ d\theta,\ \theta=\sin^{-1}x$$

$$I=\int\frac{\theta.\sin\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\ d\theta$$

$$I=\int\frac{\theta.\sin\theta}{\sqrt{\cos^2\theta}}\cos\theta\ d\theta$$

$$I=\int\theta.\sin\theta\,d\theta$$

$$Let\ u=\theta,\quad dv=\sin\theta d\theta$$

$$\therefore\ du=d\theta,\quad v=-\cos\theta d\theta$$

$$By\ \text{int}\,egration\ by\ parts,$$

$$I=\theta(-\cos\theta)-\int-\cos\theta\ d\theta$$

$$=-\theta\cos\theta+\sin\theta$$

$$=-\theta\sqrt{1-\sin^2\theta}+\sin\theta$$

$$=-\sin^{-1}x\ .\ \sqrt{1-x^2}+x$$

$$I=\int \frac{\sin^{-1}x}{\left(1-x^2\right)^{3/2}}\,dx$$

$$Put\ x=\sin\theta......(i)$$

$$dx=\cos\theta\ d\theta,\ \theta=\sin^{-1}x$$

$$I=\int\frac{\theta}{\left(1-\sin^2\theta\right)^{3/2}}\cos\theta\ d\theta$$

$$I=\int\frac{\theta}{\cos^3\theta}\cos\theta\ d\theta$$

$$I=\int\theta.\sec^2\theta\,d\theta$$

$$Let\ u=\theta,\quad dv=\sec^2\theta\ d\theta$$

$$\therefore\ du=d\theta,\quad v=\tan\theta\ d\theta$$

$$By\ \text{int}\,egration\ by\ parts,$$

$$I=\theta(\tan\theta)-\int\tan\theta\ d\theta$$

$$=\theta\tan\theta-(-\log\cos\theta)$$

$$=\theta\tan\theta+\log\cos\theta$$

$$=\sin^{-1}x.\frac{x}{\sqrt{1-x^2}}+\log\sqrt{1-x^2}$$

$$From\ (i)$$

$$\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$$

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{x}{\sqrt{1-x^2}}$$

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TECHNIQUES OF INTEGRATION:

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Let us consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where$P$ and Q are polynomials. It's possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called *proper.*

If f is improper, that is, $\deg(P) \geq \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

Where S and R are also polynomials.

Example: Find $\int \frac{x^3+x}{x-1} dx$.

Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division. This enables us to write

$$\int \frac{x^3+x}{x-1} dx. = \int (x^2 + x + 2 + \frac{2}{x-1}) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln|x-1| + C$$

INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS:

The process of expressing a rational expression and decomposing it into simpler rational expressions, that we can add or subtract to get the original rational expression, is called partial fraction decomposition.

Suppose

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated. In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

Suppose $Q(x) = (ax + b)^k$, then $\frac{R(x)}{Q(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^1} + \cdots + \frac{A_2}{(ax+b)^k}$

Suppose $Q(x) = (ax^2 + bx + c)^k$, then $\frac{R(x)}{Q(x)} = \frac{A_1x+b}{(ax^2+bx+c)} + \frac{A_2x+c}{(ax^2+bx+c)^1} + \cdots + \frac{A_2x+z}{(ax^2+bx+c)^k}$

These constants can be determined as in the following example.

Example: Evaluate$\int \frac{x^2+2x-1}{2x^3+3x^2-2x} dx$.

Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as

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$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand (2) has the form

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

To determine the values of , B, and C, we multiply both sides of this equation by the example product of the denominators, $x(2x - 1)(x + 2)$, obtaining

$$x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

$$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side—namely, 1. Likewise, the coefficients of x are equal and the constant terms are equal. This gives the following system of equations for , B, and C:

$$2A + B + 2C = 1$$

$$3A + 2B - C = 2$$

$$-2A = -1$$

Solving,

we get $A = \frac{1}{2}$, $B = 1/5$, and $C = -1/10$,

$$A = \frac{1}{2}$$

$$B + 2C = 0 \ldots (1) \qquad adding\ (1)\ \&\ (2)$$

$$2B - C = \frac{1}{2} \qquad 5B = 2 \Rightarrow B = \frac{2}{5}$$

$$4B - 2C = 2 \ldots (2) \qquad (1) \Rightarrow C = -\frac{B}{2} = -\frac{1}{5}$$

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx = \int \frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x - 1} - \frac{1}{10}\frac{1}{x + 2}\,dx$$

$$= \frac{1}{2}\,\text{In}\,|x| + \frac{1}{10}\,\ln|2x - 1| - \frac{1}{10}\,\text{In}\,|x + 2| + K$$

Example: Find$\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

The method of partial fractions gives

$$\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{A}{x - a} + \frac{B}{x + a}$$

$$1 = A(x + a) + B(x - a)$$

put $x = a$, $A(2a) = 1$. so $\text{A} = \frac{1}{2a}$. put $x = -a$, $B(-2a) = 1$, so $B = -\frac{1}{2a}$.

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int \frac{1}{x - a} - \frac{1}{x + a}\,dx$$

$$= \frac{1}{2a}(\ln|x - a| - \ln|x + a|) + C$$

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$$= \frac{1}{2a} \ln |\frac{x-a}{x+a}| + C$$

INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTION:

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Example: Find$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx$.

The first step is to divide. The result of long division is

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$$

The second step is to factor the denominator $Q(x) = x^3-x^2-x+1$. Since $Q(1) = 0$, we know that $x-1$ is a factor and we obtain

$$\begin{aligned} x^3-x^2-x+1 &= (x-1)(x^2-1) \\ &= (x-1)(x-1)(x+1) \\ &= (x-1)^2(x+1) \end{aligned}$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Put $x=1$	Put $x=-1$	Put $x=0$
$4=2B$	$-4=4C$	$0=-A+B+C$
$B=2$	$C=-1$	$0=-A+2-1$
		$A=1$

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx = \int [x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}] dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \text{In}\,|x+1| + K$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln|\frac{x-1}{x+1}| + K$$

Example: Evaluate$\int \frac{2x^2-x+4}{x^3+4x} dx$

Since $x^3+4x = x(x^2+4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$
$$2x^2 - x + 4 = A(x^2+4) + (Bx + C)x$$

Comparing the coefficient of x^2

$2 = A + B$

$B = 1$

Comparing the coefficient of x

$C = -1$

Put $x = 0$

$4 = 4A$

$A = 1$

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx = \int \frac{1}{x} + \frac{x-1}{x^2+4}\ dx$$
$$\int \frac{2x^2 - x + 4}{x(x^2 + 4)} dx = \int \frac{1}{x} dx + \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx$$
$$= \ \ln|x| + \frac{1}{2}\ \ln(x^2+4) - \frac{1}{2}\tan^{-1}(x/2) + K$$

Example: Evaluate $\int \frac{4x^2-3x+2}{4x^2-4x+3} dx$

Since the degree of the numerator is *not less than* the degree of the denominator, we first divide and obtain

$$\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} = 1 + \frac{x - 1}{4x^2 - 4x + 3}$$

Notice that the quadratic $4x^2 - 4x + 3$ is irreducible because its discriminant is $b^2 - 4ac = -32 < 0$. This means it can't be factored, so we don't need to use the partial fraction technique.

To integrate the given function we complete the square in the denominator:

$$4x^2 - 4x + 3 = (2x-1)^2 + 2$$

This suggests that we make the substitution $u = 2x - 1$. Then, $du = 2dx$ and $x = \frac{1}{2}(u+1)$, so

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} dx = \int \left(1 + \frac{x-1}{4x^2 - 4x + 3}\right) dx$$
$$= \int \left(1 + \frac{x-1}{(2x-1)^2+2}\right) dx$$
$$= x + \frac{1}{2}\int \frac{\frac{1}{2}(u+1) - 1}{u^2+2} du$$
$$= x + \frac{1}{4}\int \frac{u-1}{u^2+2} du$$
$$= x + \frac{1}{4}\int \frac{u}{u^2+2} du - \frac{1}{4}\int \frac{1}{u^2+2} du$$

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$$= x + \frac{1}{8}\ \ln\,(u^2 + 2) - \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \tan^{-1}(\frac{u}{\sqrt{2}}) + C$$

$$= x + \frac{1}{8}\ \ln\,(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}} \tan^{-1}(\frac{2x-1}{\sqrt{2}}) + C$$

Solved Problems

1 Evaluate $I = \int \frac{x^2 + x - 1}{x^3 + x^2 - 6x} dx$

Consider

$$x^3 + x^2 - 6x = x\left(x^2 + x - 6\right)$$

$$= x(x+3)(x-2)$$

$$\frac{x^2 + x - 1}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}$$

$$x^2 + x - 1 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3)$$

$$Put\ x = 0 \qquad Put\ x = 2 \qquad Put\ x = -3$$

$$-1 = -6A \qquad 5 = 10C \qquad 5 = 15B$$

$$A = \frac{1}{6} \qquad C = \frac{1}{2} \qquad B = \frac{1}{3}$$

$$\frac{x^2 + x - 1}{x(x+3)(x-2)} = \frac{1}{6}\frac{1}{x} + \frac{1}{3}\frac{1}{x+3} + \frac{1}{2}\frac{1}{x-2}$$

$$I = \frac{1}{6}\int \ \frac{1}{x}dx + \frac{1}{3}\int \ \frac{1}{x+3}dx + \frac{1}{2}\int \ \frac{1}{x-2}dx$$

$$= \frac{1}{6}\log x + \frac{1}{3}\log(x+3) + \frac{1}{2}\log(x-2)$$

.

2 Evaluate $I = \int \frac{x^2}{(x-a)(x-b)} dx$

Consider

$$\frac{x^2}{(x-a)(x-b)} = 1 + \frac{A}{x-a} + \frac{B}{x-b}$$

$$x^2 = (x-a)(x-b) + A(x-b) + B(x-a)$$

$$Put \quad x = a \qquad Put \quad x = b$$

$$a^2 = A(a-b) \quad b^2 = B(b-a)$$

$$A = \frac{a^2}{a-b} \qquad B = -\frac{b^2}{a-b}$$

$$\int \ \frac{x^2}{(x-a)(x-b)} dx = \int \ 1 + \frac{A}{x-a} dx + \int \ \frac{B}{x-b} dx$$

$$I = \int 1dx + \frac{a^2}{a-b}\int \frac{1}{x-a}dx - \frac{b^2}{a-b}\int \frac{1}{x-b}dx$$

$$I = x + \frac{a^2}{a-b}\log(x-a) - \frac{b^2}{a-b}\log(x-b)$$

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3 Evaluate $I=\int \frac{1}{x\left(x^5+1\right)}dx$

$$I=\int \frac{x^4}{x^5\left(x^5+1\right)}dx$$

(multiply Nr & Dr by x^4)

Put $x^5=z,\ then\ \ 5x^4dx=dz$

$$I=\frac{1}{5}\int \frac{1}{z\left(z+1\right)}dz$$

$$\frac{1}{z\left(z+1\right)}=\frac{A}{z}+\frac{B}{z+1}$$

$$1=A(z+1)+B(z)$$

$Put\ \ z=0 \qquad put\ \ z=-1$

$1=A \qquad\qquad B=-1$

$$\frac{1}{z\left(z+1\right)}=\frac{1}{z}-\frac{1}{z+1}$$

$$\frac{1}{5}\int \frac{1}{z\left(z+1\right)}dz=\frac{1}{5}\int \frac{1}{z}dz-\frac{1}{5}\int \frac{1}{z+1}dz$$

$$=\frac{1}{5}\left[\log z-\log(z+1)\right]$$

$$=\frac{1}{5}\left[\log x^5-\log\left(x^5+1\right)\right]$$

$$=\frac{1}{5}\log\frac{x^5}{x^5+1}$$

4 Evaluate $I=\int\limits_0^1 \frac{1}{2e^x-1}dx$

Put $e^x=z,\ then\ \ e^xdx=dz\ \ i,e.\ \ dx=\frac{dz}{z}$

$when\ x=0,\ z=e^0=1$
$when\ x=1,\ z=e$

$$I=\int\limits_0^1 \frac{1}{2e^x-1}dx$$

$$=\int\limits_1^e \frac{1}{z(2z-1)}dz$$

Consider

$$\frac{1}{z(2z-1)}=\frac{A}{z}+\frac{B}{2z-1}$$

$$1=A(2z-1)+B(z)$$

$Put\ \ z=0 \qquad Put\ \ z=\frac{1}{2}$

$A=-1 \qquad\qquad B=2$

$$\frac{1}{z(2z-1)}=\frac{-1}{z}+\frac{2}{2z-1}$$

$$\int\limits_1^e \frac{1}{z(2z-1)}dz=\int\limits_1^e \frac{-1}{z}dz+\int\limits_1^e \frac{2}{2z-1}dz$$

$$=\left[-\log z\right]_1^e+\left[\log(2z-1)\right]_1^e$$

$$=\left[-\log e+\log 1\right]+\left[\log(2e-1)-\log 1\right]$$

$$=\left[-1+\log(2e-1)\right]$$

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When the denominator contains repeated and non repeated linear factors:

5 Evaluate $I=\int\frac{x-1}{(x-2)(x+1)^2}dx$

$$\frac{x-1}{(x-2)(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-2}$$

$$x-1=A(x-2)(x+1)+B(x-2)+C(x+1)^2$$

$$Put\quad x=2\qquad Put\quad x=-1\qquad Put\quad x=0$$

$$1=9C\qquad -2=-3B\qquad -1=-2A-2B+C$$

$$C=\frac{1}{9}\qquad B=\frac{2}{3}\qquad -1=-2A-\frac{4}{3}+\frac{1}{9}$$

$$i.e.\ A=-\frac{1}{9}$$

$$\frac{x-1}{(x-2)(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-2}$$

$$I=-\frac{1}{9}\int\frac{1}{x+1}dx+\frac{2}{3}\int\frac{1}{(x+1)^2}dx+\frac{1}{9}\int\frac{1}{x-2}dx$$

$$=-\frac{1}{9}\log(x+1)+\frac{2}{3}\left[-\frac{1}{x+1}\right]+\frac{1}{9}\log(x-2)$$

6 Evaluate $I=\int\frac{3x+1}{(x+3)(x-1)^2}dx$

$$\frac{3x+1}{(x+3)(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+3}$$

$$3x+1=A(x-1)(x+3)+B(x+3)+C(x-1)^2$$

$$Put\quad x=1\qquad Put\quad x=-3\qquad Put\quad x=0$$

$$4=4B\qquad -8=16C\qquad 1=-3A+3B+C$$

$$B=1\qquad C=-\frac{1}{2}\qquad 1=-3A+3-\frac{1}{2}$$

$$i.e.\ A=\frac{1}{2}$$

$$\frac{3x+1}{(x+3)(x-1)^2}=\frac{1/2}{x-1}+\frac{1}{(x-1)^2}+\frac{-1/2}{x+3}$$

$$I=\frac{1}{2}\int\frac{1}{x-1}dx+\int\frac{1}{(x-1)^2}dx-\frac{1}{2}\int\frac{1}{x+3}dx$$

$$=\frac{1}{2}\log(x-1)-\frac{1}{x-1}-\frac{1}{2}\log(x+3)$$

The following method is useful for the case where the denominator contains repeated linear factor with high index. In this method arrange Nr and Dr in ascending powers of x.

7 Evaluate $I=\int\frac{x+1}{x^4(x-1)}dx$

Consider $\frac{1}{x^4}\frac{x+1}{-1+x}$

$$\frac{1}{x^4}\frac{x+1}{-1+x}=\frac{1}{x^4}\left(-1-2x-2x^2-2x^3\right)+\frac{2}{-1+x}$$

$$=-x^{-4}-2x^{-3}-2x^{-2}-2\frac{1}{x}+\frac{2}{-1+x}$$

$$\int\frac{1}{x^4}\frac{x+1}{-1+x}dx=\int-x^{-4}-2x^{-3}-2x^{-2}-2\frac{1}{x}+\frac{2}{-1+x}dx$$

$$=-\frac{x^{-3}}{-3}-2\frac{x^{-2}}{-2}-2\frac{x^{-1}}{-1}-2\log x+2\log(x-1)$$

$$=\frac{1}{3}\frac{1}{x^3}+\frac{1}{x^2}+2\frac{1}{x}-2\log x+2\log(x-1)$$

Divide $x+1$ by $-1+x$ till x^4 appears as a factor in the remainder.

$$\begin{array}{r|l}
 & -1-2x-2x^2-2x^3 \\ \hline
-1+x & 1+x \\
 & \underline{1-x} \\
 & \quad 2x \\
 & \quad \underline{2x-2x^2} \\
 & \qquad 2x^2 \\
 & \qquad \underline{2x^2-2x^3} \\
 & \qquad\quad 2x^3 \\
 & \qquad\quad \underline{2x^3-2x^4} \\
 & \qquad\qquad 2x^4
\end{array}$$

$$\therefore\frac{x+1}{-1+x}=\left(-1-2x-2x^2-2x^3\right)+\frac{2x^4}{-1+x}$$

TECHNIQUES OF INTEGRATION:

If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then, the expression for $R(x)/Q(x)$ will have a term of the form $\frac{cx+d}{ax^2+bx+c}$

Where A and B are constants to be determined. For instance, the function given by $f(x)=x/[(x-2)(x^2+1)(x^2+4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2+4}$$

The term can be integrated by completing the square and using the formula

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8 Evaluate $I=\int \frac{x-1}{\left(x^2+1\right)(x+1)}dx$

Consider

$$\frac{x-1}{\left(x^2+1\right)(x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

$$x-1=A\left(x^2+1\right)+(Bx+C)(x+1)$$

$Put\ x=-1$ $\quad Put\ x=0$ $\quad Compare\ coeff\ of\ x^2$

$-2=2A$ $\quad -1=A+C$ $\quad 0=A+B$

$A=-1$ $\quad C=0$ $\quad B=1$

$$\frac{x-1}{\left(x^2+1\right)(x+1)}=\frac{-1}{x+1}+\frac{x}{x^2+1}$$

$$I=\int \frac{-1}{x+1}dx+\int \frac{x}{x^2+1}dx$$

$$=-\log(x+1)+\frac{1}{2}\log\left(x^2+1\right)$$

9 Evaluate $I=\int \frac{x}{\left(x^2+4\right)(x-1)}dx$

Consider

$$\frac{x}{\left(x^2+4\right)(x-1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}$$

$$x=A\left(x^2+4\right)+(Bx+C)(x-1)$$

$Put\ x=1$ $\quad Put\ x=0$ $\quad Compare\ coeff\ of\ x^2$

$1=5A$ $\quad 0=4A-C$ $\quad 0=A+B$

$A=\frac{1}{5}$ $\quad C=\frac{4}{5}$ $\quad B=-\frac{1}{5}$

$$\frac{x}{\left(x^2+4\right)(x-1)}=\frac{1}{5}\frac{1}{x-1}-\frac{1}{5}\frac{x-4}{x^2+4}$$

$$I=\frac{1}{5}\int \frac{1}{x-1}dx-\frac{1}{5}\int \frac{x}{x^2+4}dx+\frac{4}{5}\int\frac{1}{x^2+4}dx$$

$$=\frac{1}{5}\log(x-1)-\frac{1}{5}\frac{1}{2}\log\left(x^2+4\right)+\frac{4}{5\times 2}\tan^{-1}\frac{x}{2}$$

$$=\frac{1}{5}\log(x-1)-\frac{1}{10}\log\left(x^2+4\right)+\frac{4}{10}\tan^{-1}\frac{x}{2}$$

Integration of Irrational Functions

To integrate $\int \frac{1}{quadratic}dx$

Make the coefficient of x^2 unity by taking coefficient of x^2 outside.

Complete the square in terms containing x by adding and subtracting the square of half of the coefficient of x.

Use the proper standard form.

Note: If the denominator is a perfect square, factorise and apply partial fraction method.

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1 Evaluate: $\int \frac{1}{2x^2-2x+1}dx$

$$I=\int \frac{1}{2x^2-2x+1}dx$$

$$=\frac{1}{2}\int \frac{1}{x^2-x+\frac{1}{2}}dx$$

$$=\frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+\frac{1}{2}}dx$$

$$=\int \frac{1/2}{\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}dx$$

$$=\tan^{-1}\left[\frac{x-\frac{1}{2}}{\frac{1}{2}}\right]$$

$$=\tan^{-1}(2x-1)$$

2 Evaluate: $\int \frac{1}{3x^2+12x-15}dx$

$$I=\int \frac{1}{3x^2+12x-15}dx$$

$$=\frac{1}{3}\int \frac{1}{x^2+4x-5}dx$$

$$=\frac{1}{3}\int \frac{1}{(x+2)^2-(2)^2-5}dx$$

$$=\frac{1}{3}\int \frac{1}{(x+2)^2-9}dx$$

$$=\frac{1}{3}.\frac{1}{2.3}\log\frac{(x+2)-3}{(x+2)+3}$$

$$=\frac{1}{18}\log\frac{(x+2)-3}{(x+2)+3}$$

$$Note:\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\frac{x-a}{x+a}$$

Method to evaluate $\int \frac{linear}{quadratic}dx$

Put $linear = A + B\frac{d}{dx}(quadratic)$

Equate the coefficients on both sides and find the constants and use this in the given integral and take integration.

1 Evaluate: $\int \frac{x+1}{x^2+2x+5}dx$

2 Evaluate: $\int \frac{2x+1}{3x^2+12x-15}dx$

$$I=\int\frac{x+1}{x^2+2x+5}dx$$

Let

$$x+1=A+B\frac{d}{dx}\left(x^2+2x+5\right)$$

$$x+1=A+B(2x+2)$$

Comparing the coefficients

$$1=A+5B$$

$$1=2B\ \ i.e.\ \ B=\frac{1}{2}$$

$$A+5\left(\frac{1}{2}\right)=1\quad i.e.\ \ A=1-\frac{5}{2}=-\frac{3}{2}$$

$$x+1=-\frac{3}{2}+\frac{1}{2}(2x+2)$$

$$I=\int\ \ \frac{x+1}{x^2+2x+5}dx$$

$$=\int\frac{-\frac{3}{2}+\frac{1}{2}(2x+2)}{x^2+2x+5}dx$$

$$=\int\frac{-\frac{3}{2}}{x^2+2x+5}dx+\frac{1}{2}\int\frac{(2x+2)}{x^2+2x+5}dx$$

$$=-\frac{3}{2}\int\frac{1}{(x+1)^2-1+5}dx+\frac{1}{2}\log\left(x^2+2x+5\right)$$

$$=-\frac{3}{2\times 2}\int\frac{2}{(x+1)^2+4}dx+\frac{1}{2}\log\left(x^2+2x+5\right)$$

$$=-\frac{3}{4}\tan^{-1}\left(\frac{x+1}{2}\right)+\frac{1}{2}\log\left(x^2+2x+5\right)$$

$$I=\int\frac{2x+1}{3x^2+12x-15}dx$$

Let

$$2x+1=A+B\frac{d}{dx}\left(3x^2+12x-15\right)$$

$$2x+1=A+B(6x+12)$$

comparing the coefficients

$$2=6B\ \ \ i.e.\ \ B=\frac{1}{3}$$

$$1=A+12B$$

$$1=A+12\left(\frac{1}{3}\right)\ \ i.e.\ \ A=-3$$

$$2x+1=-3+\frac{1}{3}(6x+12)$$

$$I=\int\frac{2x+1}{3x^2+12x-15}dx$$

$$=\int\frac{-3+\frac{1}{3}(6x+12)}{3x^2+12x-15}dx$$

$$=\int\frac{-3}{3x^2+12x-15}dx+\frac{1}{3}\int\frac{(6x+12)}{3x^2+12x-15}dx$$

$$=-\frac{3}{3}\int\frac{1}{x^2+4x-5}\ dx+\frac{1}{3}\int\frac{(6x+12)}{3x^2+12x-15}\ dx$$

$$=-\frac{3}{3}\int\frac{1}{(x+2)^2-4-5}\ dx+\frac{1}{3}\ \log\left(3x^2+12x-15\right)$$

$$=-\frac{3}{3}\int\frac{1}{(x+2)^2-9}\ dx+\frac{1}{3}\ \log\left(3x^2+12x-15\right)$$

$$=-1\times\frac{1}{2\times 3}\log\frac{(x+2)-3}{(x+2)+3}+\frac{1}{3}\ \log\left(3x^2+12x-15\right)$$

---

3 Evaluate $\int\frac{2x+3}{x^2+x+1}dx$

Let $I=\int\frac{2x+3}{x^2+x+1}dx$

$$2x+3=A+B\frac{d}{dx}\left(x^2+x+1\right)$$

$$=\int\frac{2}{x^2+x+1}dx+\int\frac{(2x+1)}{x^2+x+1}dx$$

$$=\int\frac{2}{\left(x+\frac{1}{2}\right)^2+1-\frac{1}{4}}dx+\log\left(x^2+x+1\right)$$

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$2x+3=A+B\left(2x+1\right)$

Comparing the coefficients

$$2x+3=A+B\left(2x+1\right)$$
$$2=2B, \qquad B=1$$
$$3=A+B, \quad A=2$$

$$I=\int\frac{2+(2x+1)}{x^2+x+1}dx$$

$$=\int\frac{2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx+\log\left(x^2+x+1\right)$$

$$=2\times\frac{2}{\sqrt{3}}\int\frac{\frac{\sqrt{3}}{2}}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx+\log\left(x^2+x+1\right)$$

$$=\frac{4}{\sqrt{3}}\tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}+\log\left(x^2+x+1\right)$$

4 Evaluate: $\displaystyle\int\frac{\left(x-1\right)^2}{x^2+2x+2}dx$

Let $\displaystyle I=\int\frac{x^2-2x+1}{x^2+2x+2}dx$

$$\begin{array}{r|l} & \qquad\qquad 1 \\ \hline x^2+2x+2 & x^2-2x+1 \\ & \underline{x^2+2x+2} \\ & \qquad -4x-1 \end{array}$$

5 Evaluate: $\displaystyle\int_0^1\frac{4x^2+3}{8x^2+4x+5}dx$

$$\begin{array}{r|l} & \quad 1/2 \\ \hline 8x^2+4x+5 & 4x^2+0x+3 \\ & \underline{4x^2+2x+5/2} \\ & \quad -2x+1/2 \end{array}$$

$$I=\int 1-\frac{4x+1}{x^2+2x+2}dx$$

$$=x-\int\frac{4x+1}{x^2+2x+2}dx$$

Consider

$$4x+1=A+B\frac{d}{dx}\left(x^2+2x+2\right)$$

$$4x+1=A+B(2x+2)$$

Comparing the coefficients

$$4=2B \quad i.e. \quad B=2$$

$$1=A+2B$$

$$1=A+4 \quad i.e. \quad A=-3$$

$$4x+1=-3+2(2x+2)$$

$$I=x-\int\frac{-3+2(2x+2)}{x^2+2x+2}dx$$

$$=x-\left[\int\frac{-3}{x^2+2x+2}dx+2\int\frac{(2x+2)}{x^2+2x+2}dx\right]$$

$$=x+3\int\frac{1}{(x+1)^2-1+2}dx-2\int\frac{(2x+2)}{x^2+2x+2}dx$$

$$=x+3\int\frac{1}{(x+1)^2+1}dx-2\int\frac{(2x+2)}{x^2+2x+2}dx$$

$$=x+3\tan^{-1}(x+1)-2\log\left(x^2+2x+2\right)$$

$$I=\int_0^1 \frac{1}{2}+\frac{-2x+\frac{1}{2}}{8x^2+4x+5}dx$$

$$=\frac{1}{2}-\frac{1}{2}\int_0^1\frac{4x-1}{8x^2+4x+5}dx$$

Consider $4x-1=A+B\frac{d}{dx}\left(8x^2+4x+5\right)$

$$4x-1=A+B(16x+4)$$

Comparing the coefficients: $4=16B \quad i.e. \quad B=\frac{1}{4}$

$$-1=A+4B, \quad -1=A+1 \quad i.e. \quad A=-2$$

$$4x-1=-2+\tfrac{1}{4}(16x+4)$$

$$I=\frac{1}{2}-\frac{1}{2}\int_0^1\frac{-2+\frac{1}{4}(16x+4)}{8x^2+4x+5}dx$$

$$=\frac{1}{2}-\frac{1}{2}\left[\int_0^1\frac{-2}{8x^2+4x+5}dx+\frac{1}{4}\int_0^1\frac{(16x+4)}{8x^2+4x+5}dx\right]$$

$$=\frac{1}{2}+\frac{2}{2\times 8}\int_0^1\frac{1}{x^2+\frac{1}{2}x+\frac{5}{8}}dx-\frac{1}{8}\int_0^1\frac{(16x+4)}{8x^2+4x+5}dx$$

$$=\frac{1}{2}+\frac{1}{8}\int_0^1\frac{1}{\left(x+\frac{1}{4}\right)^2-\frac{1}{16}+\frac{5}{8}}dx-\frac{1}{8}\left[\log\left(8x^2+4x+5\right)\right]_0^1$$

$$=\frac{1}{2}+\frac{1}{8}\int_0^1\frac{1}{\left(x+\frac{1}{4}\right)^2+\frac{9}{16}}-\frac{1}{8}\left[\log 17-\log 5\right]$$

$$=\frac{1}{2}+\frac{1}{8}\frac{4}{3}\left[\tan^{-1}\frac{\left(x+\frac{1}{4}\right)}{\frac{3}{4}}\right]_0^1-\frac{1}{8}\log\frac{17}{5}$$

$$=\frac{1}{2}+\frac{1}{6}\left[\tan^{-1}\left(\frac{5}{3}\right)-\tan^{-1}\left(\frac{1}{3}\right)\right]-\frac{1}{8}\log\frac{17}{5}$$

To evaluate $\int\sqrt{quadratic}\ dx \ \ or \ \int\frac{1}{\sqrt{quadratic}}\ dx$

Make the coefficient of x^2 unity by taking its coefficient outside the square root sign.
Complete the square in terms containing x by adding and subtracting the square of half of the coefficient of x.

Use proper standard form

$\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$ $\int\sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}$ $\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}$	$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$ $\int\frac{dx}{\sqrt{a^2+x^2}}=\sinh^{-1}\frac{x}{a}$ $\int\frac{dx}{\sqrt{x^2-a^2}}=\cosh^{-1}\frac{x}{a}$ $\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}$

.

1 Evaluate $\int\sqrt{x^2-2x+2}\ dx$

$$\int\sqrt{x^2-2x+2}\ dx=\int\sqrt{(x-1)^2-1^2+2}\ dx$$
$$=\int\sqrt{(x-1)^2+1^2}\ dx$$
$$=\frac{x}{2}\sqrt{(x-1)^2+1^2}+\frac{1^2}{2}\sinh^{-1}\frac{(x-1)}{1}$$

2 Evaluate $\int\sqrt{3+2x-x^2}\ dx$

$$\int\sqrt{3+2x-x^2}\ dx=\int\sqrt{3-\left(x^2-2x\right)}\ dx$$
$$=\int\sqrt{3+1^2-(x-1)^2}\ dx$$
$$=\int\sqrt{2^2-(x-1)^2}\ dx$$
$$=\frac{x}{2}\sqrt{2^2-(x-1)^2}+\frac{2^2}{2}\sin^{-1}\frac{(x-1)}{2}$$

.

3 Evaluate $\int\frac{1}{\sqrt{x^2-2x+2}}\ dx$

$$\int\frac{1}{\sqrt{x^2-2x+2}}\ dx=\int\frac{1}{\sqrt{(x-1)^2-1^2+2}}\ dx$$
$$=\int\frac{1}{\sqrt{(x-1)^2+1^2}}\ dx$$
$$=\sinh^{-1}(x-1)$$

4 Evaluate $\int\frac{1}{\sqrt{3+2x-x^2}}\ dx$

$$\int\frac{1}{\sqrt{3+2x-x^2}}\ dx=\int\frac{1}{\sqrt{3-\left(x^2-2x\right)}}\ dx$$
$$=\int\frac{1}{\sqrt{3+1^2-(x-1)^2}}\ dx$$
$$=\int\frac{1}{\sqrt{2^2-(x-1)^2}}\ dx$$
$$=\sin^{-1}\frac{(x-1)}{2}$$

.

5 Evaluate $\int\frac{1}{\sqrt{3x-2-x^2}}\ dx$

$$\int\frac{1}{\sqrt{3x-2-x^2}}\ dx=\int\frac{1}{\sqrt{-2-\left(x^2-3x\right)}}\ dx$$

$$=\int \frac{1}{\sqrt{-2+\frac{9}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx$$

$$=\int \frac{1}{\sqrt{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx$$

$$=\sin^{-1}\frac{\left(x-\frac{3}{2}\right)}{\frac{1}{2}}$$

$$=\sin^{-1}\left(2x-3\right)$$

Method to evaluate $\int linear\sqrt{quadratic}\ dx \quad or \quad \int \frac{linear}{\sqrt{quadratic}}\,dx$

Put $linear = A + B\frac{d}{dx}(quadratic)$

Equate the coefficients on both sides and find the constants and use this in the given integral and take integration.

1 **Evaluate** $I=\int (x+2)\sqrt{2x^2+2x+1}\ dx$

$$(x+2)=A+B\frac{d}{dx}\left(2x^2+2x+1\right)$$

$$x+2=A+B(4x+2)$$

$$comparing\ the\ coefficients$$

$$1=4B \qquad 2=A+2B$$

$$B=\frac{1}{4} \qquad 2=A+\frac{2}{4}\ \ i.e.\ A=\frac{3}{2}$$

$$I=\int\left[\frac{3}{2}+\frac{1}{4}(4x+2)\right]\sqrt{2x^2+2x+1}\ dx$$

$$I=\frac{3}{2}\int\sqrt{2x^2+2x+1}\ dx+\frac{1}{4}\int\ (4x+2)\sqrt{2x^2+2x+1}\ dx$$

$$I=\frac{3}{2}\sqrt{2}\int\sqrt{x^2+x+\frac{1}{2}}\ dx+\frac{2}{4}\sqrt{2}\int\ (2x+1)\sqrt{x^2+x+\frac{1}{2}}\ dx$$

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In the second integral, let $u = x^2 + x + \frac{1}{2}$

$$du = (2x+1)dx$$

$$I = \frac{3}{\sqrt{2}}\int\sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}+\frac{1}{2}}\;dx + \frac{2\sqrt{2}}{4}\int \sqrt{u}\;du$$

$$I = \frac{3}{\sqrt{2}}\int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}\;dx + \frac{1}{\sqrt{2}}\int u^{\frac{1}{2}}\;du$$

$$= \frac{3}{\sqrt{2}}\left[\frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}+\frac{\frac{1}{4}}{2}\sinh^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{1}{2}}\right]+\frac{1}{\sqrt{2}}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \frac{3}{\sqrt{2}}\left[\frac{1}{2}\frac{2x+1}{2}\frac{1}{2}\sqrt{(2x+1)^2+1}+\frac{1}{8}\sinh^{-1}(2x+1)\right]+\frac{2}{3\sqrt{2}}\left(x^2+x+\frac{1}{2}\right)^{\frac{3}{2}}$$

$$= \frac{3}{8\sqrt{2}}(2x+1)\sqrt{(2x+1)^2+1}+\frac{3}{8\sqrt{2}}\sinh^{-1}(2x+1)+\frac{2}{3\sqrt{2}}\frac{1}{2\sqrt{2}}\left(2x^2+2x+1\right)^{\frac{3}{2}}$$

$$= \frac{3}{8\sqrt{2}}(2x+1)\sqrt{(2x+1)^2+1}+\frac{3}{8\sqrt{2}}\sinh^{-1}(2x+1)+\frac{1}{6}\left(2x^2+2x+1\right)^{\frac{3}{2}}$$

.

2 **Evaluate** $I = \int\frac{2x+5}{\sqrt{x^2-2x+2}}\,dx$

$$(2x+5) = A + B\frac{d}{dx}\left(x^2-2x+2\right)$$

$$2x+5 = A + B(2x-2)$$

comparing the coefficients

$2 = 2B$ $\qquad 5 = A - 2B$

$B = 1$ $\qquad A = 7$

3 **Evaluate** $I = \int\frac{(x+2)}{\sqrt{2x^2+2x+1}}\,dx$

$$(x+2) = A + B\frac{d}{dx}\left(2x^2+2x+1\right)$$

$$x+2 = A + B(4x+2)$$

comparing the coefficients

$1 = 4B$ $\qquad 2 = A + 2B$

$B = \frac{1}{4}$ $\qquad 2 = A + \frac{2}{4}$ *i.e.* $A = \frac{3}{2}$

$$I = \int \frac{2x+5}{\sqrt{x^2-2x+2}}\,dx = \int \frac{7+(2x-2)}{\sqrt{x^2-2x+2}}\,dx$$

$$I = 7\int \frac{1}{\sqrt{x^2-2x+2}}\,dx + \int \frac{2x-2}{\sqrt{x^2-2x+2}}\,dx$$

$$I = 7\int \frac{1}{\sqrt{(x-1)^2-1+2}}\,dx + \int \frac{2x-2}{\sqrt{x^2-2x+2}}\,dx$$

In the second integral, let $u = x^2-2x+2$

$$du = (2x-2)dx$$

$$I = 7\int \frac{1}{\sqrt{(x-1)^2+1}}\,dx + \int \frac{1}{\sqrt{u}}\,du$$

$$I = 7\sinh^{-1}(x-1) + \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]$$

$$= 7\sinh^{-1}(x-1) + 2\sqrt{u}$$

$$= 7\sinh^{-1}(x-1) + 2\sqrt{x^2-2x+2}$$

$$I = \int \frac{(x+2)}{\sqrt{2x^2+2x+1}}\,dx$$

$$= \int \frac{\frac{3}{2}+\frac{1}{4}(4x+2)}{\sqrt{2x^2+2x+1}}\,dx$$

$$= \frac{3}{2}\int \frac{1}{\sqrt{2x^2+2x+1}}\,dx + \frac{1}{4}\int \frac{(4x+2)}{\sqrt{2x^2+2x+1}}\,dx$$

In the second integral, let

$$u = 2x^2+2x+1,\ du = (4x+2)dx$$

$$= \frac{3}{2\sqrt{2}}\int \frac{1}{\sqrt{x^2+x+\frac{1}{2}}}\,dx + \frac{1}{4}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{3}{2\sqrt{2}}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+\frac{1}{2}}}\,dx + \frac{1}{4}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{3}{2\sqrt{2}}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}}\,dx + \frac{1}{4}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{3}{2\sqrt{2}}\sinh^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{1}{2}} + \frac{1}{4}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]$$

$$= \frac{3}{2\sqrt{2}}\sinh^{-1}(2x+1) + \frac{2}{4}\sqrt{2x^2+2x+1}$$

4. Evaluate $I = \int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx$

$$x^2+2x+3 = A\left(x^2+x+1\right) + B\frac{d}{dx}\left(x^2+x+1\right) + C$$

$$x^2+2x+3 = A\left(x^2+x+1\right) + B\left(2x+1\right) + C$$

Comparing the coefficients of x^2, x, *constant term*

$1 = A$ $\qquad 2 = A+2B$ *i.e.* $B = \frac{1}{2}$ $\qquad 3 = A+B+C$ *i.e.* $C = \frac{3}{2}$

$$I = \int \frac{x^2 + 2x + 3}{\sqrt{x^2 + x + 1}} dx$$

$$= \int \frac{\left(x^2 + x + 1\right) + \frac{1}{2}\left(2x + 1\right) + \frac{3}{2}}{\sqrt{x^2 + x + 1}} dx$$

$$= \int \frac{\left(x^2 + x + 1\right)}{\sqrt{x^2 + x + 1}} dx + \frac{1}{2}\int \frac{\left(2x + 1\right)}{\sqrt{x^2 + x + 1}} dx + \frac{3}{2}\int \frac{1}{\sqrt{x^2 + x + 1}} dx$$

$$= \int \sqrt{x^2 + x + 1}\, dx + \frac{1}{2}\int \frac{\left(2x + 1\right)}{\sqrt{x^2 + x + 1}}\, dx + \frac{3}{2}\int \frac{1}{\sqrt{x^2 + x + 1}}\, dx$$

In the sec*ond* int*egral*, *let* $u = x^2 + x + 1, \quad du = (2x + 1)dx$

$$= \int \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1}\, dx + \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du + \frac{3}{2}\int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1}}\, dx$$

$$= \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}\, dx + \frac{1}{2}\int u^{-\frac{1}{2}}\, du + \frac{3}{2}\int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}}\, dx$$

$$= \left[\frac{\left(x + \frac{1}{2}\right)}{2}\sqrt{x^2 + x + 1} + \frac{\frac{3}{4}}{2}\sinh^{-1}\frac{\left(x + \frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} \right] + \frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right] + \frac{3}{2}\sinh^{-1}\frac{\left(x + \frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}$$

$$= \left[\frac{2x + 1}{4}\sqrt{x^2 + x + 1} + \frac{3}{8}\sinh^{-1}\frac{\left(2x + 1\right)}{\sqrt{3}} \right] + \sqrt{x^2 + x + 1} + \frac{3}{2}\sinh^{-1}\frac{\left(2x + 1\right)}{\sqrt{3}}$$

IMPROPER INTEGRALS:

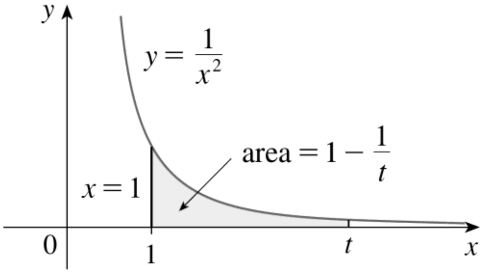
In defining a definite integral $\int_a^b f\,(x)dx$ we dealt with a function f defined on a finite interval $[a, b]$ and we assumed that f does not have an infinite discontinuity. In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an *improper* integral. One of the most important applications of this idea is probability distributions.

TYPE I: INFINITE INTERVALS

Consider the infinite region A that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$. One may think that, since A is infinite in extent, its area must be infinite, but let's take a closer look. The area of the part of S that lies to the left of the line $x = t$ is

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}]_1^t = 1 - \frac{1}{t}$$

Notice that $A(t) < 1$ no matter how large t is chosen.



We also observe that

$$\lim_{t\to\infty} A\,(t) = \lim_{t\to\infty} (\,1 - \frac{1}{t}) = 1$$

The area of the shaded region approaches 1 as $\to \infty$, so we say that the area of the infinite region A is equal to 1 and we write

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = 1$$

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE I

(a) If $\int_a^t f\,(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f\,(x)dx = \lim_{t\to\infty} \int_a^t f\,(x)dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f\,(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f\,(x)dx = \lim_{t\to-\infty} \int_t^b f\,(x)dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f\,(x)dx$ and $\int_{-\infty}^b f\,(x)dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If both $\int_a^\infty f\,(x)dx$ and $\int_{-\infty}^a f\,(x)dx$ are convergent, then we define

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$$\int_{-\infty}^{\infty} f\,(x)dx = \int_{-\infty}^{a} f\,(x)dx + \int_{a}^{\infty} f\,(x)dx$$

In part (c) any real number a can be used.

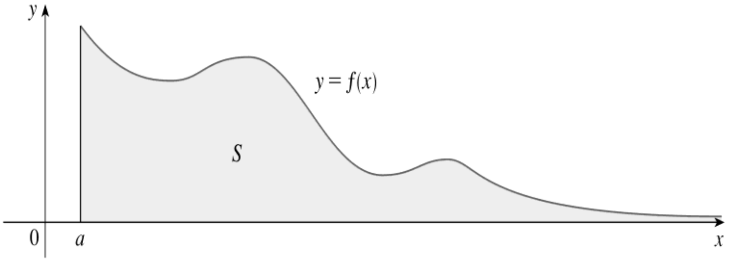
Note:

1 The above integrals convergent if the associated limit exists and is a finite number and divergent if the associated limit either doesn't exist or it is infinity.

2 Any of the above improper integrals can be interpreted as an area provided that f is a positive function. For instance, in case (a) if $f(x) \geq 0$ and the integral $\int_a^{\infty} f\,(x)dx$ is convergent, then we define the area of the region $S = \{(x, y) | x \geq a, 0 \leq y \leq f(x)\}$

$$A(S) = \int_{a}^{\infty} f\,(x)dx$$

This is appropriate because $\int_a^{\infty} f\,(x)dx$ is the limit as $t \to \infty$ of the area under the graph of f from a to t.



Determine whether the integral $\int_1^{\infty}(\,1/x)dx$ is convergent or divergent.

According to Definition(a), we have

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x} dx$$

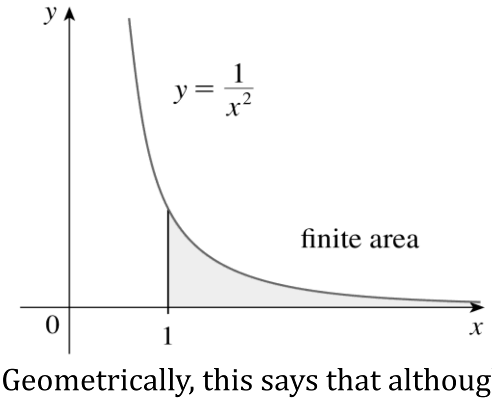
$$= \lim_{t\to\infty} \ln\, |x|]_1^t$$

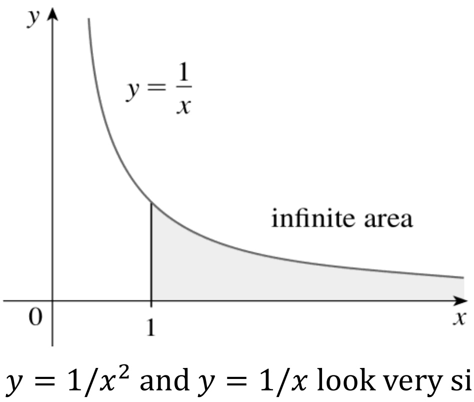
$$= \lim_{t\to\infty} (In\, t - \ln 1)$$

$$= \lim_{t\to\infty} \text{In}\, t = \infty$$

The limit does not exist as a finite number and so the improper integral $\int_1^{\infty}(\,1/x)dx$ is divergent.

EXAMPLE: $\int_1^{\infty} \frac{1}{x^2} dx$**converges** $\int_1^{\infty} \frac{1}{x} dx$ **diverges**





Geometrically, this says that although the curves $y = 1/x^2$ and $y = 1/x$ look very similar for $x > 0$, the region under $y = 1/x^2$ to the right of $x = 1$ has finite area whereas the

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corresponding region under $y = 1/x$ has infinite area.

Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$ but $1/x^2$ approaches 0 faster than $1/x$. The values of $1/x$ don't decrease fast enough for its integral to have a finite value.

Solved Problems

1 Evaluate $\int_{-\infty}^{0} x\, e^x dx$.

Using part (b) of Definition 1, we have

$$\int_{-\infty}^{0} x\, e^X dx = \lim_{t\to-\infty} \int_t^0 x\, e^x dx$$

$$= \lim_{t\to-\infty} [xe^x]_t^0 - \lim_{t\to-\infty} \int_t^0 e^x\, dx$$

$$= \lim_{t\to-\infty} [0 - te^t] - \lim_{t\to-\infty} [e^x]_t^0$$

$$= \lim_{t\to-\infty} [0 - te^t] - \lim_{t\to-\infty} [1 - e^t]$$

$$= -0 - 1 + 0 = -1$$

We integrate by parts with
$u = x, \quad dv = e^x dx$ so that
$du = dx, \quad v = e^x$

We know that $e^t \to 0$ as $t \to -\infty$, and by L'Hospital's Rule we have

$$\lim_{t\to-\infty} t\, e^t = \lim_{t\to-\infty} \frac{t}{e^{-t}}$$

$$= \lim_{t\to-\infty} \frac{1}{-e^{-t}}$$

$$= \lim_{t\to-\infty} (-e^t) = 0$$

2 Show that $\int_0^1 \frac{1}{x} dx$ diverges

The function $\frac{1}{x}$ is continuous on $(0,1]$ and unbounded near 0, and we have

$$\int_0^1 \frac{1}{x} dx = \underset{c\to 0^+}{Lt} \int_c^1 \frac{1}{x} dx$$

$$= \underset{c\to 0^+}{Lt} [\ln x]_c^1$$

$$= \underset{c\to 0^+}{Lt} [\ln 1 - \ln c]$$

$$= 0 - \ln 0$$

$$= \infty$$

Therefore the given integral diverges

3 Show that $\int_1^\infty \frac{1}{x^2} dx$ converges

$$\int_1^\infty \frac{1}{x^2} dx = \underset{c\to\infty}{Lt} \int_1^c \frac{1}{x^2} dx$$

$$= \underset{c\to\infty}{Lt} \left[-\frac{1}{x}\right]_1^c$$

$$= \underset{c\to\infty}{Lt} \left[1 - \frac{1}{c}\right]$$

$$= 1$$

Therefore $\int_1^\infty \frac{1}{x^2} dx$ converges.

Note:

1 To show that $\int_0^\infty f(x) dx$ diverges, it is enough to find one d in $(0,\infty)$ for which $\int_0^d f(x)$ or

$\int_{d}^{\infty} f(x)dx$ diverges.

2 But $\int_{0}^{\infty} f(x)dx$ converges, both $\int_{0}^{d} f(x)dx$ and $\int_{d}^{\infty} f(x)dx$ converges where d is in $(0,\infty)$.

.

4 Show that $\int_{0}^{\infty} \frac{1}{x^2}dx$ is divergent

This is an integral over an infinite interval and it also contains a discontinuous integrand. Hence we split the integral into two as follows:

$$\int_{0}^{\infty} \frac{1}{x^2}dx = \int_{0}^{1} \frac{1}{x^2}dx + \int_{1}^{\infty} \frac{1}{x^2}dx$$

First let us evaluate $\int_{0}^{1} \frac{1}{x^2}dx$

The function $\frac{1}{x^2}$ is continuous on $(0,1]$ and unbounded near 0, and we have

$$\int_{0}^{1} \frac{1}{x^2}dx = \underset{c \to 0^+}{Lt} \int_{c}^{1} \frac{1}{x^2}dx$$

$$= \underset{c \to 0^+}{Lt} \left[-\frac{1}{x}\right]_c^1$$

$$= \underset{c \to 0^+}{Lt} \left[-1+\frac{1}{c}\right]$$

$$= \infty$$

Since the first integral is divergent, by Note 1, given integral is divergent.

5 Evaluate $\int_{1}^{\infty} \frac{dx}{x^{3/2}}$

6 Evaluate $\int_{1}^{\infty} \frac{dx}{\sqrt{x}}$

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$$\int_1^\infty \frac{dx}{x^{3/2}} = \underset{t\to\infty}{Lt}\int_1^t \frac{1}{x^{3/2}}dx$$

$$= \underset{t\to\infty}{Lt}\int_1^t x^{-3/2}dx$$

$$= \underset{t\to\infty}{Lt}\left[\frac{x^{-1/2}}{-1/2}\right]_1^t$$

$$= \underset{t\to\infty}{Lt}\left[-\frac{2}{\sqrt{x}}\right]_1^t$$

$$= \underset{t\to\infty}{Lt}\left[-\frac{2}{\sqrt{t}}+2\right]$$

$$= 2$$

The integral is convergent

$$\int_1^\infty \frac{dx}{x^{1/2}} = \underset{t\to\infty}{Lt}\int_1^t \frac{1}{x^{1/2}}dx$$

$$= \underset{t\to\infty}{Lt}\int_1^t x^{-1/2}dx$$

$$= \underset{t\to\infty}{Lt}\left[\frac{x^{1/2}}{1/2}\right]_1^t$$

$$= \underset{t\to\infty}{Lt}\left[2\sqrt{x}\right]_1^t$$

$$= \underset{t\to\infty}{Lt}\left[2\sqrt{t}-2\right]$$

$$= \infty$$

The integral is divergent

▪

7 Evaluate $\int_0^\infty \frac{4a}{x^2+4a^2}dx$

$$\int_0^\infty \frac{4a}{x^2+4a^2}dx = \underset{t\to\infty}{Lt}\,2\int_0^t \frac{2a}{x^2+(2a)^2}dx$$

$$= \underset{t\to\infty}{Lt}\,2\left[\tan^{-1}\frac{x}{2a}\right]_0^t$$

$$= \underset{t\to\infty}{Lt}\,2\left[\tan^{-1}\frac{t}{2a}-0\right]$$

$$= 2\,\tan^{-1}\infty$$

$$= 2\frac{\pi}{2}$$

The integral is convergent

8 Evaluate $\int_1^\infty \frac{x}{(1+x)^3}dx$

$$\int_1^\infty \frac{x}{(1+x)^3}dx = \underset{t\to\infty}{Lt}\int_0^t \frac{(1+x)-1}{(1+x)^3}dx$$

$$= \underset{t\to\infty}{Lt}\int_0^t \frac{(1+x)}{(1+x)^3}dx - \underset{t\to\infty}{Lt}\int_0^t \frac{1}{(1+x)^3}dx$$

$$= \underset{t\to\infty}{Lt}\int_0^t (1+x)^{-2}dx - \underset{t\to\infty}{Lt}\int_0^t (1+x)^{-3}dx$$

$$= \underset{t\to\infty}{Lt}\left[\frac{(1+x)^{-1}}{-1}\right]_0^t - \underset{t\to\infty}{Lt}\left[\frac{(1+x)^{-2}}{-2}\right]_0^t$$

$$= \underset{t\to\infty}{Lt}\left[-\frac{1}{1+x}\right]_0^t - \underset{t\to\infty}{Lt}\left[-\frac{1}{2(1+x)^2}\right]_0^t$$

$$= \underset{t\to\infty}{Lt}\left[-\frac{1}{1+t}+1\right] - \underset{t\to\infty}{Lt}\left[-\frac{1}{2(1+t)^2}+\frac{1}{2}\right]$$

$$= 1-\frac{1}{2}$$

The integral is convergent

▪

9 Determine whether the improper integral $\int_0^3 \frac{dx}{\sqrt{x}}$ converges or diverges.

(the integrand is infinite at lower limit)

$$\int_0^3 \frac{dx}{x^{1/2}} = \underset{\in\to 0}{Lt} \int_{\in}^{3} x^{-1/2} dx$$

$$= \underset{\in\to 0}{Lt} \left[\frac{x^{1/2}}{1/2} \right]_{\in}^{3}$$

$$= \underset{\in\to 0}{Lt} \left[2\sqrt{3} - 2\sqrt{\in} \right]$$

$$= 2\sqrt{3}$$

The integral is convergent

10 Evaluate $\int_0^1 \frac{dx}{x^2}$

(the integrand is infinite at lower limit)

$$\int_0^1 \frac{dx}{x^2} = \underset{\in\to 0}{Lt} \int_{\in}^{1} x^{-2} dx$$

$$= \underset{\in\to 0}{Lt} \left[\frac{x^{-1}}{-1} \right]_{\in}^{1}$$

$$= \underset{\in\to 0}{Lt} \left[-\frac{1}{x} \right]_{\in}^{1}$$

$$= \underset{\in\to 0}{Lt} \left[-1 + \frac{1}{\in} \right] \to \infty$$

The integral is divergent

.

11 Evaluate $\int_0^\infty x\, e^{-x^2} dx$

$$\int_0^\infty x\, e^{-x^2} dx = \underset{t\to\infty}{Lt} \int_0^t x\, e^{-x^2} dx$$

$Put\ z = x^2,\ dz = 2x\ dx$

$when\ \ x = 0,\ z = 0\ \ \&\ \ when\ x = t,\ z = t^2$

$$\int_0^\infty x\, e^{-x^2} dx = \underset{t\to\infty}{Lt}\ \frac{1}{2} \int_0^{t^2} e^{-z} dz$$

$$= \underset{t\to\infty}{Lt}\ \frac{1}{2} \left[\frac{e^{-z}}{-1} \right]_0^{t^2}$$

$$= \underset{t\to\infty}{Lt}\ \frac{1}{2} \left[-e^{-t^2} + 1 \right]$$

$$= \frac{1}{2} \left[-e^{-\infty} + 1 \right]$$

$$= \frac{1}{2}$$

12 Evaluate $\int_0^\infty \frac{1}{(1+x)\sqrt{x}} dx$

$$\int_1^{\infty} \frac{1}{(1+x)\sqrt{x}} dx = \underset{t \to \infty}{Lt} \int_1^{t} \frac{1}{(1+x)\sqrt{x}} dx$$

$$Put \;\; z = \sqrt{x},\; dz = \frac{1}{2\sqrt{x}} dx,\; z^2 = x$$

$$when \; x = 1,\, t = 1;\; when \; x = t,\, z = \sqrt{t}$$

$$\int_1^{\infty} \frac{1}{(1+x)\sqrt{x}} dx = \underset{t \to \infty}{Lt} \; 2 \int_1^{\sqrt{t}} \frac{1}{(1+z^2)} dz$$

$$= \underset{t \to \infty}{Lt} \; 2 \left[\tan^{-1} z \right]_1^{\sqrt{t}}$$

$$= \underset{t \to \infty}{Lt} \; 2 \left[\tan^{-1} \sqrt{t} - \tan^{-1} 1 \right]$$

$$= 2 \left[\tan^{-1} \infty - \tan^{-1} 1 \right]$$

$$= 2 \left[\frac{\pi}{2} - \frac{\pi}{4} \right]$$

$$= \frac{\pi}{2}$$

.

13 Prove that $\int_a^{\infty} \frac{1}{x^n} dx,\; a > 0$ **converges if and only if** $n > 1$**.**

$$\int_a^{\infty} \frac{1}{x^n} dx = \underset{t \to \infty}{Lt} \int_a^{t} \frac{1}{x^n} dx$$

Case I : When $n = 1$

$$\int_a^{\infty} \frac{1}{x^n} dx = \underset{t \to \infty}{Lt} \int_a^{t} \frac{1}{x} dx$$

$$= \underset{t \to \infty}{Lt} \left[\log x \right]_a^t$$

$$= \underset{t \to \infty}{Lt} \left[\log t - \log a \right]$$

$$= \log \infty - \log a = \infty$$

Case II : When $n \neq 1$

14 Prove that $\int_a^{b} \frac{1}{(b-x)^n} dx$ **converges iff** $n < 1$**.** (the integrand is infinite at upper limit)

$$\int_a^{b} \frac{1}{(b-x)^n} dx = \underset{\in \to 0}{Lt} \int_a^{b-\in} \frac{1}{(b-x)^n} dx$$

Case I : When $n = 1$

$$\int_a^{b} \frac{1}{(b-x)} dx = \underset{\in \to 0}{Lt} \int_a^{b-\in} \frac{1}{(b-x)} dx$$

$$= \underset{\in \to 0}{Lt} \left[-\log(b-x) \right]_a^{b-\in}$$

$$= \underset{\in \to 0}{Lt} \left[-\log \in + \log(b-a) \right]$$

$$= -\log 0 + \log(b-a) = \infty$$

Case IL : When $n \neq 1$

$$\int_a^\infty \frac{1}{x^n}dx = \underset{t\to\infty}{Lt}\int_a^t x^{-n}dx$$

$$= \underset{t\to\infty}{Lt}\left[\frac{x^{1-n}}{1-n}\right]_a^t$$

$Sub\,case\ (i)\ when\ n<1, 1-n>0$

$$\int_a^\infty \frac{1}{x^n}dx = \underset{t\to\infty}{Lt}\frac{1}{1-n}\left[t^{1-n}-a^{1-n}\right]$$

$$=\frac{1}{1-n}\left[\infty-a^{1-n}\right]=\infty$$

$Sub\,case\ (ii)\ when\ n>1,\ i.e.\ n-1>0$

$$\int_a^\infty \frac{1}{x^n}dx=\underset{t\to\infty}{Lt}\left[\frac{x^{1-n}}{1-n}\right]_a^t$$

$$=\underset{t\to\infty}{Lt}\left[-\frac{1}{n-1}\frac{1}{x^{n-1}}\right]_a^t$$

$$=\underset{t\to\infty}{Lt}\frac{-1}{n-1}\left[\frac{1}{t^{n-1}}-\frac{1}{a^{n-1}}\right]$$

$$=-\frac{1}{n-1}\left[\frac{1}{\infty}-\frac{1}{a^{n-1}}\right]$$

$$=\frac{1}{n-1}\frac{1}{a^{n-1}}$$

$$\int_a^b \frac{1}{(b-x)^n}dx=\underset{\in\to 0}{Lt}\int_a^{b-\in}(b-x)^{-n}dx$$

$$=\underset{\in\to 0}{Lt}\left[\frac{(b-x)^{1-n}}{-(1-n)}\right]_a^{b-\in}$$

$$=\frac{1}{n-1}\underset{\in\to 0}{Lt}\left[\in^{1-n}-(b-a)^{1-n}\right]$$

$Sub\,case\ (i)\ when\ n<1, 1-n>0$

$$\int_a^b \frac{1}{(b-x)^n}dx=\underset{\in\to 0}{Lt}\frac{-1}{1-n}\left[\in^{1-n}-(b-a)^{1-n}\right]$$

$$=\frac{1}{1-n}\left[(b-a)^{1-n}\right]=finite$$

$Sub\,case\ (ii)\ when\ n>1,\ i.e.\ n-1>0$

$$\int_a^b \frac{1}{(b-x)^n}dx=\underset{\in\to 0}{Lt}\frac{1}{n-1}\left[\frac{1}{\in^{n-1}}-\frac{1}{(b-a)^{n-1}}\right]$$

$$=\frac{1}{n-1}\left[\infty-\frac{1}{(b-a)^{n-1}}\right]=\infty$$

15 Determine whether the integral $\int_0^\infty \frac{1}{x^2+4}dx$ is convergent or divergent.

$$\int_0^\infty \frac{1}{x^2+4}dx=\underset{t\to\infty}{Lt}\ \frac{1}{2}\int_0^t \frac{2}{x^2+(2)^2}dx$$

$$=\underset{t\to\infty}{Lt}\ \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^t$$

$$=\underset{t\to\infty}{Lt}\ \frac{1}{2}\left[\tan^{-1}\frac{t}{2}-\tan^{-1}0\right]$$

∴ The integral is convergent

$$=\frac{1}{2}\tan^{-1}\infty$$

$$=\frac{1}{2}.\frac{\pi}{2}$$

.

16 Determine whether $\int_{-\infty}^{\infty} xe^{-x^2}dx$ is convergent or divergent.

Since both the limits are infinite, we split the integral such that each integral has one infinite limit as follows:

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx$$

Consider $\int_{-\infty}^{0} xe^{-x^2}dx = \underset{t\to-\infty}{Lt}\int_{t}^{0} xe^{-x^2}dx$

$$= \underset{t\to-\infty}{Lt}\int_{t}^{0} \frac{1}{2}e^{-x^2}d\left(x^2\right)$$

$$= \underset{t\to-\infty}{Lt} -\frac{1}{2}\left[e^{-x^2}\right]_{t}^{0}$$

$$= \underset{t\to-\infty}{Lt} -\frac{1}{2}\left[1-e^{-t^2}\right]$$

$$= -\frac{1}{2} \quad \left[\sin ce\ e^{-\infty}=0\right]$$

Consider $\int_{0}^{\infty} xe^{-x^2}dx = \underset{t\to\infty}{Lt}\int_{0}^{t} xe^{-x^2}dx$

$$= \underset{t\to\infty}{Lt}\int_{0}^{t} \frac{1}{2}e^{-x^2}d\left(x^2\right)$$

$$= \underset{t\to\infty}{Lt} -\frac{1}{2}\left[e^{-x^2}\right]_{0}^{t}$$

$$= \underset{t\to\infty}{Lt} -\frac{1}{2}\left[e^{-t^2}-1\right]$$

$$= \frac{1}{2} \quad \left[\sin ce\ e^{-\infty}=0\right]$$

Therefore $\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx = -\frac{1}{2}+\frac{1}{2}=0$

17. Evaluate$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$.

It’s convenient to choose $a = 0$:

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_{0}^{\infty}\frac{1}{1+x^2}dx$$

We must now evaluate the integrals on the right side separately:

$$\int_{0}^{\infty}\frac{1}{1+x^2}dx = \lim_{t\to\infty}\int_{0}^{t}\frac{dx}{1+x^2}$$

$$= \lim_{t\to\infty}\tan^{-1}x]_0^t$$

$$= \lim_{t\to\infty}(\tan^{-1}t - \tan^{-1}0)$$

$$= \lim_{t\to\infty}\tan^{-1}t$$

$$= \frac{\pi}{2}$$

$$\int_{-\infty}^{0}\frac{1}{1+x^2}dx = \lim_{t\to-\infty}\int_{t}^{0}\frac{dx}{1+x^2}$$

$$= \lim_{t\to-\infty}\tan^{-1}x]_t^0$$

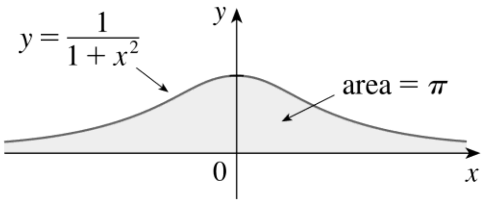
$$= \lim_{t\to-\infty}(\tan^{-1}0 - \tan^{-1}t)$$

$$= 0 - \left(-\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}$$

Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$



Note: Since $1/(1+x^2) > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the curve $y = 1/(1+x^2)$ and above the x-axis

18 For what values of p is the integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ convergent?

If $p = 1$, then it is proved that the integral $\int_1^{\infty} \frac{1}{x}\,dx$ is divergent.

let's assume that $p > 1$. Then	
$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^{t} x^{-p}\,dx$	
$= \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}]_{x=1}^{x=t}$	If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.
$= \lim_{t\to\infty} \frac{1}{1-p}[\frac{1}{t^{p-1}} - 1]$	
$= \lim_{t\to\infty} \frac{1}{1-p}[0 - 1]$	
$= \frac{1}{p-1}$	∴ If $p > 1$, the integral converges.
let's assume that $p < 1$. Then	
$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^{t} x^{-p}\,dx$	
$= \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}]_{x=1}^{x=t}$	if $p < 1$, then $p - 1 < 0$ and so
$= \lim_{t\to\infty} \frac{1}{1-p}[\frac{1}{t^{p-1}} - 1]$	$\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$
$= \lim_{t\to\infty} \frac{1}{1-p}[\propto - 1]$	
$= \propto$	∴ If $p < 1$, the integral diverges.

$\int_1^{\infty} \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \le 1$.

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TYPE 2: DISCONTINUOUS INTEGRANDS

Suppose that f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b. Let S be the unbounded region under the graph of f and above the x-axis between $a$and b. (For Type 1 integrals, the regions extended indefinitely in a.

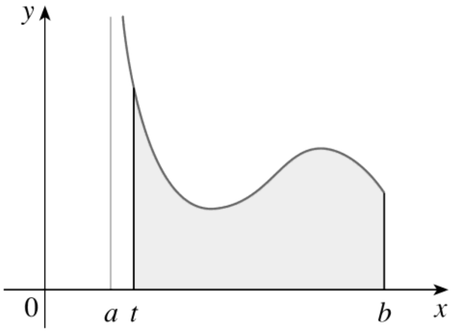
DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2:

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$ and f has $vertical$ asymptotes at b, if this limit exists .

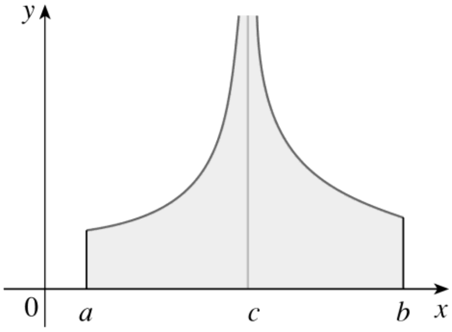
(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$ and f has $vertical$ asymptotes at a, if this limit exists .



(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



Example: Find$\int_2^5 \frac{1}{\sqrt{x-2}} dx$.

The given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$. Therefore by definition,

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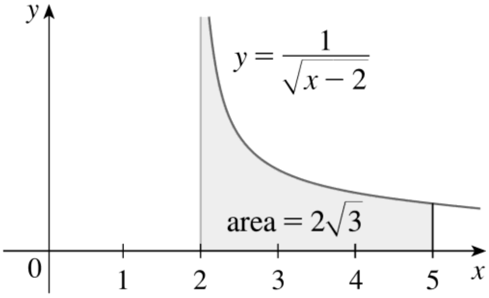
$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}}$$

$$= \lim_{t\to 2^+} 2\sqrt{x-2}]_t^5$$

$$= \lim_{t\to 2^+} 2\,(\sqrt{3}-\sqrt{t-2})$$

$$= 2\sqrt{3}$$



Thus the given improper integral is convergent and, since the integrand is positive, we can interpret the value of the integral as the area of the shaded region in Figure.

Example: Determine whether $\int_0^{\pi/2}$ sec xdx converges or diverges.

The given integral is improper because $\lim_{t\to\frac{\pi}{2}} \sec x = \infty$. Therefore by definition

$$\int_0^{\frac{\pi}{2}} \sec x dx = \lim_{t\to(\pi/2)^-} \int_0^t \sec\ x dx$$

$$= \lim_{t\to(\pi/2)^-} \ln\ |\sec x + \tan x|]_0^t$$

$$= \lim_{t\to(\pi/2)^-} [\ \ln\ (\sec t + \tan t) \text{ - } ln\ 1] = \infty$$

because $\sec t \to \infty$ and $\tan t \to \infty$ as $t \to (\pi/2)^-$.
Thus the given improper integral is divergent.

Example: Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

Observe that the line $x = 1$ is a vertical asymptote of the integrand. Since it occurs in the middle of the interval [0, 3]. Therefore by definition

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

Where

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$$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1}$$

$$= \lim_{t\to 1-} \ln |x-1|]_0^t$$

$$= \lim_{t\to 1-} (\ln|t-1| - \ln|-1|)$$

$$= \lim_{t\to 1-} \ln (1-t) = -\infty, \quad log0 = -\infty$$

Thus $\int_0^1 d\,x/(x-1)$ is divergent.

This implies that $\int_0^3 d\,x/(x-1)$ is divergent. [We do not need to evaluate $\int_1^3 d\,x/(x-1)$

If we had not noticed the asymptote $x = 1$ in the above problem, then we might have made the following erroneous calculation:

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|]_0^3 = \ln 2 - \ln 1 = \ln 2$$

This is wrong because the integral is improper and must be calculated in terms of limits.

Example: Show that $\int_{-2}^{3} \frac{1}{x^3} dx$ is divergent

This integrand is not continuous at x = 0 and hence split the integral as follows:

$$\int_{-2}^{3} \frac{1}{x^3} dx = \int_{-2}^{0} \frac{1}{x^3} dx + \int_{0}^{3} \frac{1}{x^3} dx$$

Consider the first integral $\int_{-2}^{0} \frac{1}{x^3} dx = \underset{t\to 0-}{Lt} \int_{-2}^{t} \frac{1}{x^3} dx$

$$= \underset{t\to 0-}{Lt} \left[-\frac{1}{2x^2} \right]_{-2}^{t}$$

$$= \underset{t\to 0-}{Lt} \left[-\frac{1}{2t^2} + \frac{1}{8} \right]$$

$$= -\infty$$

Since the first integral is divergent, the entire integral is divergent.

A COMPARISON TEST FOR IMPROPER INTEGRALS

Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent. In such cases the following theorem will be useful. Although we state it for Type1 integrals, a similar theorem is true for Type 2

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integrals.

COMPARISON THEOREM:

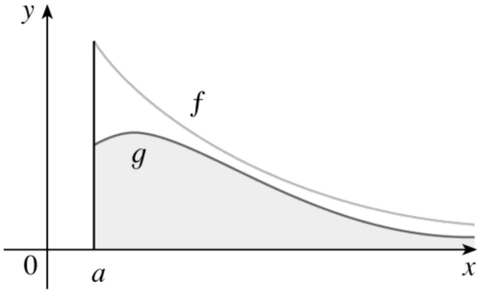
Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f\,(x)dx$ is convergent, then $\int_a^\infty g\,(x)dx$ is convergent.

(b) If $\int_a^\infty g\,(x)dx$ is divergent, then $\int_a^\infty f\,(x)dx$ is divergent.

This is explained in the following Figure. If the area under the top curve $y = f(x)$ is finite, then so is the area under the bottom curve $= g(x)$. And if the area under $y = g(x)$ is infinite, then so is the area under $f(x)$.

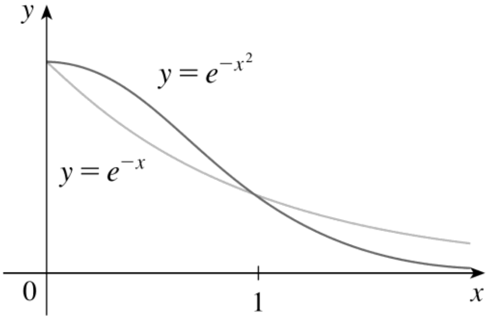
[Note that the reverse is not necessarily true: If $\int_a^\infty g\,(x)dx$ is convergent, $\int_a^\infty f\,(x)dx$ may or may not be convergent, and if $\int_a^\infty f\,(x)dx$ is divergent, $\int_a^\infty g\,(x)dx$ may or may not be divergent.]



Example: Show that $\int_0^\infty e^{-x^2}\,dx$ is convergent.

We can't evaluate the integral directly because the antiderivative of e^{-x^2} is not an elementary function. We write

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$



and observe that the first integral on the right-hand side is just an ordinary definite integral. In the second integral we use the fact that for $x \geq 1$ we have $x^2 \geq x$, so $-x^2 \leq -x$ and therefore $e^{-x^2} \leq e^{-x}$. (See Figure .) The integral of e^{-x} is easy to evaluate:

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$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}(e^{-1} - e^{-t}) = e^{-1}$$

Therefore by comparison theorem, $\int_1^\infty e^{-x^2}\,dx$ is convergent.

Also $\int_0^1 e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^1 = 1 - e^{-1}$

Therefore by comparison theorem, $\int_0^1 e^{-x^2}\,dx$ is convergent.

therefore $\int_0^\infty e^{-x^2}\,dx$ is convergent

Example: show that by comparison test, $\int_1^\infty \frac{e^{-x}}{x}dx$ is convergent.

To prove the integral is convergent, we need a larger integrand. From the given limit

$$\begin{aligned} &x > 1 \\ i.e.\ &\frac{1}{x} < 1 \\ &\frac{e^{-x}}{x} < e^{-x}, \qquad \text{Since } e^{-x} > 0 \end{aligned}$$

Hence $\int_1^\infty \frac{e^{-x}}{x}dx < \int_1^\infty e^{-x}\,dx$ …….(1)

Consider $\int_1^\infty e^{-x}\,dx = \underset{t\to\infty}{Lt}\int_1^t e^{-x}\,dx$

$$= \underset{t\to\infty}{Lt}\left[\frac{e^{-x}}{-1}\right]_1^t$$

$$= \underset{t\to\infty}{Lt}\left[-e^{-t} + e^{-1}\right]$$

$=\left[e^{-1}\right]$ and hence convergent.

Hence from (1) $\int_1^\infty \frac{e^{-x}}{x}dx$ is convergent

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Example: show that by comparison test, $\int_1^\infty e^{-x^2}\ dx$ is convergent.

To prove the integral is convergent, we need a larger integrand.
We know that e^{-x} is decreasing function and hence whenever $x_1 > x_2\ \ then\ e^{-x_1} < e^{-x_2}$

From the given limit, we have $x > 1\ \ i.e.\ x^2 > x\ \ and\ hence\ e^{-x^2} < e^{-x}$. .

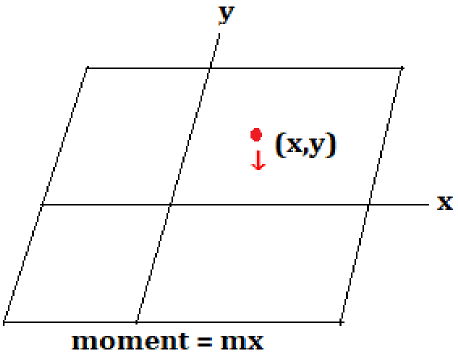
Hence $\int_1^\infty e^{-x^2}\ dx < \int_1^\infty e^{-x}\ dx$ ....(1)

But we know that $\int_1^\infty e^{-x}\ dx = e^{-1}$, which is convergent and hence from (1) , by comparison test

we conclude that $\int_1^\infty e^{-x^2}\ dx$ is convergent.

Moments and Centres of Gravity

Moment is a quantity which measures the tendency of mass to produce rotation. It is used to define a point called the centre of gravity of a set of points in a plane. The moment of the point mass about the $y-$axis is defined to be mx.



Suppose that there are several point masses whose masses are $m_1, m_2, \ldots m_n$ located at the respective points $(x_1, y_1), (x_2, y_2), \ldots\ldots (x_n, y_n)$ in a plane. The moment M_y of the collection of all point masses about the $y-$axis is the sum of moments of all point masses. $\therefore$
$M_y = m_1x_1 + \ldots\ldots + m_nx_n$
If $M_y = 0$, then the point masses is said to be in equilibrium.

Similarly, we can define the moment of point masses $m_1, m_2, \ldots m_n$ about the $x-$ axis by setting $M_x = m_1y_1 + \ldots. + m_ny_n$.

Now let $m = m_1 + m_2 + \ldots. + m_n$ be the combined mass of all point masses considered and let us take a point mass with mass m at $(\bar{x}, \bar{y})$, then its moment about the x axis and y axis is M_x

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and M_y respectively.

Now $m\bar{x} = M_y = m_1x_1 + ... + m_nx_n$ and $m\bar{y} = M_x = m_1y_1 + ... + m_ny_n$.

Therefore $\bar{x} = \frac{M_y}{m} = \frac{m_1x_1 + ... + m_nx_n}{m}$ and $\bar{y} = \frac{M_x}{m} = \frac{m_1y_1 + ... + m_ny_n}{m}$

The point $(\bar{x}, \bar{y})$ is called the centre of gravity or centroid of the given collection of point masses.

Find the moments and centre of gravity of objects with masses 2, 3, 6 and 8 located ate the points (2,1), (–1,3), (3,–2) and (3,0) respectively.

The moments are

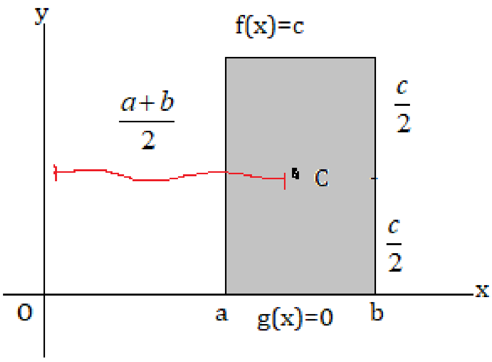
$M_y = m_1x_1 + ... + m_nx_n = 2(2) + 3(-1) + 6(3) + 8(3) = 43$ and

$M_x = m_1y_1 + ... + m_ny_n = 2(1) + 3(3) + 6(-2) + 8(0) = -1$

Since $m = 2 + 3 + 6 + 8 = 19$, we have $\bar{x} = \frac{M_y}{m} = \frac{43}{19}$ and $\bar{y} = \frac{M_x}{m} = \frac{-1}{19}$

Definition: Let f and g be continuous on $[a,b]$, with $g(x) \le f(x)$ for $a \le x \le b$ and let R be the region between the graphs of f and g on $[a,b]$. Then the moment M_x of R about the x axis is given by $M_x = \frac{1}{2}\int_a^b [f(x)]^2 - [g(x)]^2 dx$ and the moment M_y of R about the y axis is given by $M_y = \int_a^b x[f(x) - g(x)]dx$.

Note: If R has positive area A, then the centre of mass or centre of gravity of R is the point $(\bar{x}, \bar{y})$ defined by $\bar{x} = \frac{M_y}{A}$ and $\bar{y} = \frac{M_x}{A}$.



1 Find the moment about the y axis of the rectangle R bounded by the lines $x = a$, $x = b$, $y = 0$ and $y = c$.

Here we have $f(x) = c$ and $g(x) = 0$. Therefore the height of the rectangle is c. So

$$M_y = \int_a^b x[f(x) - g(x)]dx$$

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$$= \int_a^b x[c-0]dx$$

$$= c\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{c}{2}\left(b^2 - a^2\right)$$

$$= c(b-a)\frac{b+a}{2}$$

2 Let R be the semicircular region bounded by the y axis and the graphs of $f(x) = \sqrt{r^2 - x^2}$ and $g(x) = -\sqrt{r^2 - x^2}$ for $0 \le x \le r$. Find the moments.

The moment M_x of R about the x axis is

$$M_x = \frac{1}{2}\int_a^b [f(x)]^2 - [g(x)]^2\, dx$$

$$M_x = \frac{1}{2}\int_0^r \left(r^2 - x^2\right) - \left(r^2 - x^2\right) dx = 0$$

The moment M_y of R about the y axis is

$$M_y = \int_a^b x[f(x) - g(x)]dx$$

$$= 2\int_0^r x\sqrt{r^2 - x^2}\, dx$$

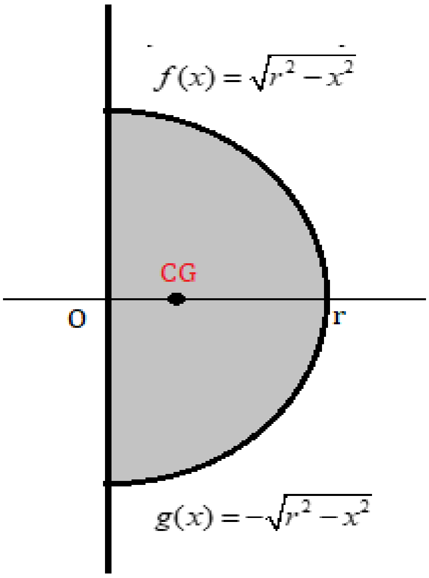
$$= -\int_{r^2}^0 \sqrt{u}\, du$$

$$= \int_0^{r^2} u^{\frac{1}{2}} du$$

$$= \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{r^2}$$

$$= \frac{2}{3}r^3$$

Since the area A of R is $\frac{1}{2}\pi r^2$, we have



Let $r^2 - x^2 = u$. Then $-2xdx = du$
when $x = 0,\ u = r^2$
when $x = r,\ u = 0$

$$\overline{x} = \frac{M_y}{A} = \frac{\frac{2r^3}{3}}{\frac{\pi r^2}{2}} = \frac{4r}{3\pi} \quad \text{and} \quad \overline{y} = \frac{M_x}{A} = 0$$

MASS AND CENTRE OF MASS

3. Find the mass M and the centre of mass $\overline{x}$ of a rod lying on the $x-$ axis over the interval [1,2] whose density function is given by $\delta(x) = 2 + 3x^2$

We know that Mass

$$M = \int_a^b \delta(x)dx$$

$$= \int_1^2 \left(2 + 3x^2\right)dx$$

$$= \left[2x + x^3\right]_1^2$$

$$= (4+8) - (2+1)$$

$$= 9$$

Therefore center of mass

$$\overline{x} = \frac{M_0}{M} = \frac{\int_a^b x.\delta(x)\ dx}{\int_a^b \delta(x)dx}$$

$$\int_a^b x.\delta(x)\ dx = \int_1^2 x.\left(2 + 3x^2\right)\ dx$$

$$= \left[x^2 + \frac{3x^4}{4}\right]_1^2$$

$$= \left(4 + 12\right) - \left(1 + \frac{3}{4}\right)$$

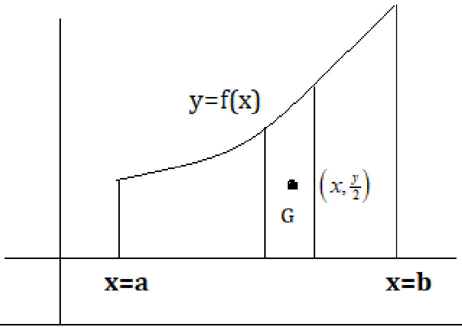
$$= 16 - \frac{7}{4}$$

$$= \frac{57}{4}$$

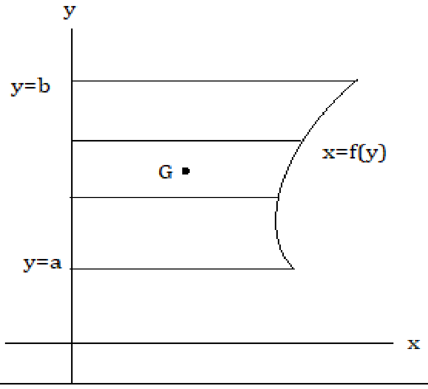
$$\overline{x} = \frac{M_0}{M} = \frac{\frac{57}{4}}{9} = \frac{19}{12}$$

A lamina is a thin flat sheet having uniform thickness. The centre of gravity of a lamina is the point where it balances perfectly, i.e. the lamina's centre of mass.

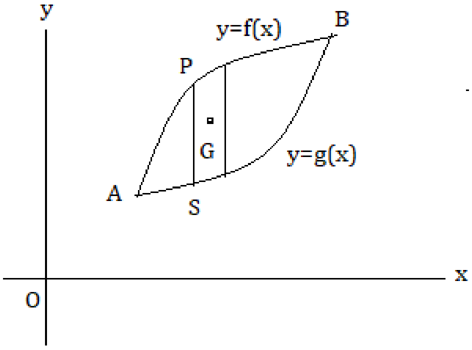
Centre of gravity of plane area bounded by the curve $y = f(x)$, the x axis and the ordinates $x = a$ and $x = b$.

$\bar{x}=\frac{\int_a^b x(ydx)}{\int_a^b ydx}$	$\bar{y}=\frac{\int_a^b \frac{y}{2}(ydx)}{\int_a^b ydx}$ If the area is symmetrical about x axis, then $\bar{y}=0$.	

Centre of gravity of plane area bounded by the curve $x=f(y)$, the y axis and the abscissae $y=a$ and $y=b$.

$\bar{x}=\frac{\int_a^b \frac{x}{2}(xdy)}{\int_a^b xdy}$ If the area is symmetrical about y axis, then $\bar{x}=0$.	$\bar{y}=\frac{\int_a^b y(xdy)}{\int_a^b xdy}$	

Cerntre of gravity of the area enclosed between two curves

$\bar{x}=\frac{\int x(y_1-y_2)dx}{\int (y_1-y_2)dx}$ The limit of integration points of intersection	$\bar{y}=\frac{\int \frac{y_1+y_2}{2}(y_1-y_2)dx}{\int (y_1-y_2)dx}$ being the values of x for the A and B	 Here $P(x,y_1)$ and $Q(x,y_2)$

Centre of gravity of the sectorial area bounded by the curve $r=f(\theta)$ and the radii vectors $\theta=\alpha$ and $\theta=\beta$.

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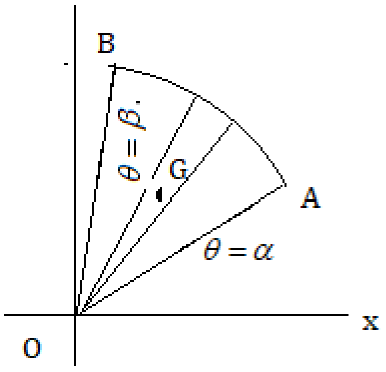
$\bar{x} = \dfrac{\frac{2}{3}\int_{\alpha}^{\beta} r^3 \cos\theta d\theta}{\int_{\alpha}^{\beta} r^2 d\theta}$	$\bar{y} = \dfrac{\frac{2}{3}\int_{\alpha}^{\beta} r^3 \sin\theta d\theta}{\int_{\alpha}^{\beta} r^2 d\theta}$	

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4 Find the position of the centroid of the area bounded by the curve $y = 5x - x^2$ and the x-axis.

Let $C(\bar{x}, \bar{y})$ be the coordinates of the centroid. Then $\bar{x} = \dfrac{\int_a^b x(ydx)}{\int_a^b ydx}$

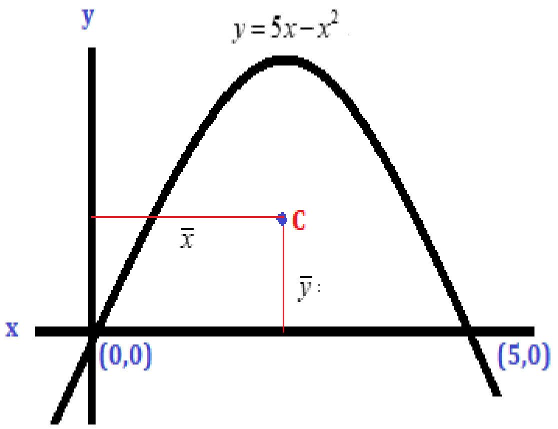
$$= \frac{\int_0^5 x(5x - x^2)dx}{\int_0^5 (5x - x^2)dx}$$

$$= \frac{\left[5\frac{x^3}{3} - \frac{x^4}{4}\right]_0^5}{\left[5\frac{x^2}{2} - \frac{x^3}{3}\right]_0^5}$$

$$= \frac{\frac{625}{3} - \frac{625}{4}}{\frac{125}{2} - \frac{125}{3}}$$

$$= \frac{\frac{625}{12}}{\frac{125}{6}}$$

$$= \frac{5}{2}$$



$$\bar{y} = \frac{\int_0^5 \frac{y}{2}(y)dx}{\int_0^5 (y)dx}$$

$$= \frac{\int_0^5 \frac{(5x - x^2)}{2}(5x - x^2)dx}{\int_0^5 (5x - x^2)dx}$$

$$= \frac{\frac{1}{2}\int_0^5 (25x^2 + x^4 - 10x^3)dx}{\frac{125}{6}}, \quad \textit{from } \bar{x}$$

$$= \frac{6}{250}\left[\frac{25x^3}{3} + \frac{x^5}{5} - \frac{10x^4}{4}\right]_0^5$$

$$= \frac{6}{250}\left[\frac{3125}{3} + \frac{3125}{5} - \frac{6250}{4}\right]$$

$$= \frac{5}{2}$$

.

5 Find the position of the centroid of the area bounded by the curve $y=3x^2$, the x-axis and the ordinates x = 0 and x = 2.

Let $C(\bar{x},\bar{y})$ be the coordinates of the centroid.

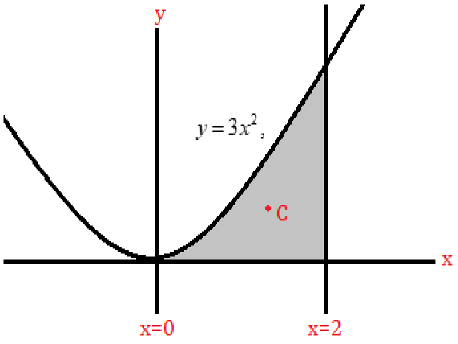
Then $$\bar{x} = \frac{\int_a^b x(ydx)}{\int_a^b ydx}$$

$$= \frac{\int_0^2 x(3x^2)dx}{\int_0^2 (3x^2)dx}$$

$$= \frac{\left[\frac{3x^4}{4}\right]_0^2}{\left[\frac{3x^3}{3}\right]_0^2}$$

$$= \frac{\left[\frac{48}{4}\right]}{\left[\frac{24}{3}\right]}$$

$$= \frac{3}{2}$$



$$\bar{y} = \frac{\int_0^2 \frac{y}{2}(y)dx}{\int_0^2 (y)dx}$$

$$= \frac{\int_0^2 \frac{(3x^2)}{2}(3x^2)dx}{\int_0^2 (3x^2)dx}$$

$$= \frac{\frac{1}{2}\int_0^2 (9x^4)dx}{\frac{24}{3}}, \quad from\ \bar{x}$$

$$= \frac{9}{16}\left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{9}{16}\left[\frac{32}{5}\right]$$

$$= \frac{18}{5}$$

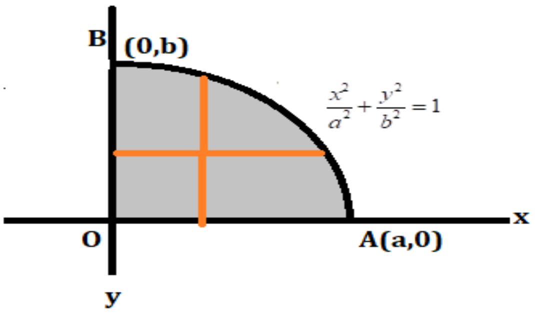
6 Find the centre of gravity of a plane lamina of uniform density in the form of a quadrant of an ellipse.

Let the equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The parametric equations are
$x=a\cos t, \qquad y=b\sin t$

$dx=-a\sin t dt, \qquad dy=b\cos t\, dt$

Here θ varies from 0 to $\frac{\pi}{2}$.



$$\overline{x}=\frac{\int_a^b x\left(ydx\right)}{\int_a^b ydx}$$

$$=\frac{\int_0^{\frac{\pi}{2}} a\cos t.b\sin t.(-a\sin t)dt}{\int_0^{\frac{\pi}{2}} b\sin t.(-a\sin t)dt}$$

$$=\frac{a\int_0^{\frac{\pi}{2}}\cos t.\sin^2 tdt}{\int_0^{\frac{\pi}{2}}\sin^2 tdt}$$

$$=\frac{a.\frac{1}{3.1}}{\frac{1}{2}.\frac{\pi}{2}}$$

$$=\frac{4a}{3\pi}$$

$$\overline{y}=\frac{\int_a^b \frac{y}{2}\left(ydx\right)}{\int_a^b ydx}$$

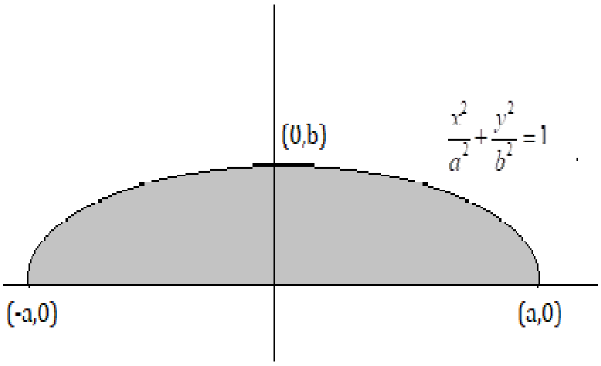
$$=\frac{\frac{1}{2}\int_{\frac{\pi}{2}}^{0} b^2\sin^2 t.(-a\sin t)dt}{\int_{\frac{\pi}{2}}^{0} b\sin t.(-a\sin t)dt}$$

$$=\frac{\frac{b}{2}\int_0^{\frac{\pi}{2}}\sin^3 tdt}{\int_0^{\frac{\pi}{2}}\sin^2 tdt}$$

$$=\frac{\frac{b}{2}\frac{2}{3.1}}{\frac{1}{2}.\frac{\pi}{2}}$$

$$=\frac{4b}{3\pi}$$

7 Find the centre of gravity of a plane lamina bounded by x axis and the part of the ellipse for which y is positive.



Let the equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The parametric equations are
$x=a\cos t,\qquad y=b\sin t$

$dx=-a\sin tdt,\qquad dy=b\cos t\,dt$
Here θ varies from 0 to π.

$$\overline{y}=\frac{\int_{-a}^{a}\frac{y}{2}\left(ydx\right)}{\int_{-a}^{a} ydx}$$

$$=\frac{\frac{1}{2}\int_0^{\pi} b^2\sin^2 t.(-a\sin t)dt}{\int_0^{\pi} b\sin t.(-a\sin t)dt}$$

Since the area is symmetric about y axis.

Hence

$$\overline{x}=\frac{\int_{-a}^{a}x(ydx)}{\int_{-a}^{a}ydx}=0$$

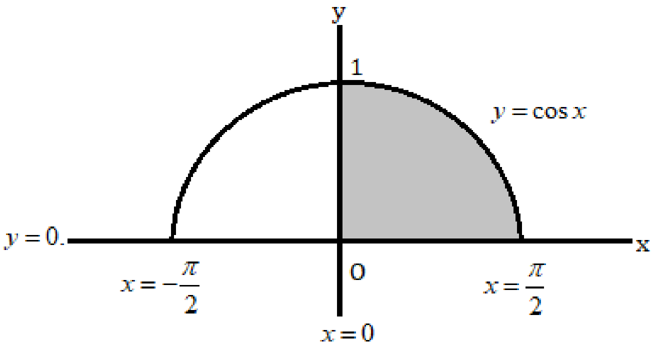
$$=\frac{\frac{b}{2}.2\int_{0}^{\frac{\pi}{2}}\sin^3 tdt}{2\int_{0}^{\frac{\pi}{2}}\sin^2 tdt}$$

$$=\frac{b.\frac{2}{3.1}}{2.\frac{1}{2}.\frac{\pi}{2}}$$

$$=\frac{4b}{3\pi}$$

.

8 Find the centre of gravity of the area between the curve $y=\cos x$ from $x=0$ to $x=-\frac{\pi}{2}$ bounded by the line $y=0.$



The equation of the curve is $y=\cos x$

$$\overline{x}=\frac{\int_{0}^{\frac{\pi}{2}}x(ydx)}{\int_{0}^{\frac{\pi}{2}}ydx}$$

$$=\frac{\int_{0}^{\frac{\pi}{2}}x\cos xdx}{\int_{0}^{\frac{\pi}{2}}\cos xdx}$$

$$=\frac{[(x)(\sin x)-(1)(-\cos x)]_0^{\frac{\pi}{2}}}{[\sin x]_0^{\frac{\pi}{2}}}$$

$$\overline{y}=\frac{\int_{0}^{\frac{\pi}{2}}\frac{y}{2}(ydx)}{\int_{0}^{\frac{\pi}{2}}ydx}$$

$$=\frac{\frac{1}{2}\int_{0}^{\frac{\pi}{2}}\cos^2 xdx}{\int_{0}^{\frac{\pi}{2}}\cos xdx}$$

$$=\frac{\frac{1}{4}\int_{0}^{\frac{\pi}{2}}1+\cos 2xdx}{\int_{0}^{\frac{\pi}{2}}\cos xdx}$$

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$$=\frac{\left[x\sin x+\cos x\right]_0^{\frac{\pi}{2}}}{\left[\sin x\right]_0^{\frac{\pi}{2}}}$$

$$=\frac{\frac{\pi}{2}.1+0-1}{1-0}$$

$$=\frac{\pi}{2}-1$$

$$=\frac{\frac{1}{4}\left[x+\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}}{\left[\sin x\right]_0^{\frac{\pi}{2}}}$$

$$=\frac{\frac{1}{4}\left[\frac{\pi}{2}\right]}{1}=\frac{\pi}{8}$$

.

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9 Find the centroid of the cardioid
$r=a(1+\cos\theta)$

Since the curve is symmetrical about the initial line $\theta=0$, $\overline{y}=0$.

$$\overline{x}=\frac{\frac{2}{3}\int_{\alpha}^{\beta} r^3\cos\theta d\theta}{\int_{\alpha}^{\beta} r^2 d\theta}$$

$$Nr=\frac{2}{3}\int_{-\pi}^{\pi} r^3\cos\theta d\theta$$

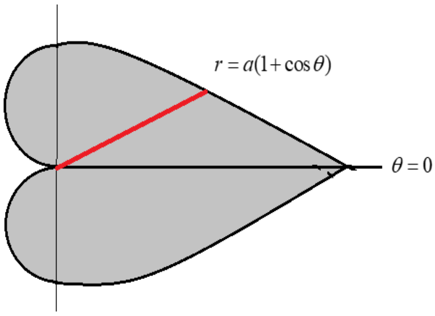
$$=2\times\frac{2}{3}\int_{0}^{\pi} r^3\cos\theta d\theta$$

$$=2\times\frac{2}{3}\int_{0}^{\pi} a^3(1+\cos\theta)^3\cos\theta d\theta$$

$$=\frac{4}{3}a^3\int_{0}^{\pi}\left[\cos\theta+\cos^4\theta+3\cos^2\theta+3\cos^3\theta\right]d\theta$$

$$But\ \int_{0}^{\pi}\cos^n\theta d\theta=0\ when\ n\ is\ odd$$

$$Also\ \int_{0}^{\pi}\cos^n\theta d\theta=2\int_{0}^{\pi/2}\cos^n\theta d\theta\ when\ n\ is\ even$$



$$Dr=\int_{-\pi}^{\pi} r^2 d\theta$$

$$=2\times\int_{0}^{\pi} r^2 d\theta$$

$$=2\times a^2\int_{0}^{\pi}(1+\cos\theta)^2 d\theta$$

$$=2a^2\int_{0}^{\pi}1+\cos^2\theta+2\cos\theta\ d\theta$$

$$=2a^2\left[\left[\theta\right]_0^{\pi}+2(0)+2\int_{0}^{\frac{\pi}{2}}\cos^2\theta\ d\theta\right]$$

$$=2a^2\left[\pi+2\frac{1}{2}\frac{\pi}{2}\right]$$

$$=2\pi a^2\left[1+\frac{1}{2}\right]$$

$$=3\pi a^2$$

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$$= a^3 \frac{4}{3} \times 2\int_0^{\frac{\pi}{2}} \left[\cos^4\theta + 3\cos^2\theta\right] d\theta$$

$$= \frac{8}{3}\times a^3 \left[\frac{3.1}{4.2}\frac{\pi}{2} + 3.\frac{1}{2}\frac{\pi}{2}\right]$$

$$= \frac{8}{3}\times a^3 \left[\frac{3.1}{4.2}\frac{\pi}{2} + 3.\frac{1}{2}\frac{\pi}{2}\right]$$

$$= \frac{8}{3}\times a^3 \times \frac{\pi}{4}\left[\frac{15}{4}\right]$$

$$= \frac{5}{2}\pi a^3$$

Therefore $\bar{x} = \dfrac{\frac{5}{2}\pi a^3}{3\pi a^2} = \dfrac{5a}{6}$

i.e. $(\bar{x},\bar{y}) = \left(\dfrac{5a}{6},0\right)$

10 Find the centroid of the area of a loop of the lemniscate $r^2 = a^2\cos 2\theta$

Since the curve is symmetrical about the initial line. Therefore $\bar{y} = 0$.

$$\bar{x} = \frac{\frac{2}{3}\int_{\alpha}^{\beta} r^3\cos\theta d\theta}{\int_{\alpha}^{\beta} r^2 d\theta} = \frac{\frac{2}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^3\cos\theta d\theta}{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 d\theta}$$

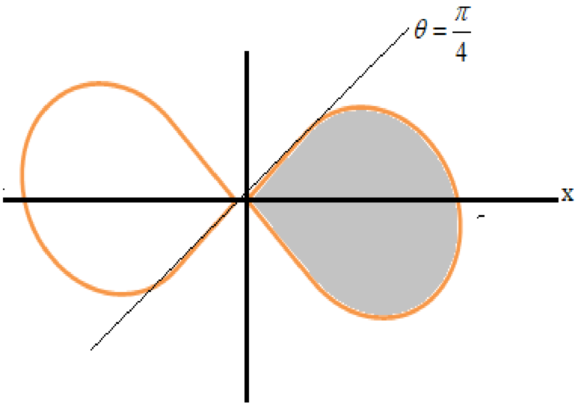
$$Nr = \frac{2}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^3\cos\theta d\theta$$

$$= 2.\frac{2}{3}\int_0^{\frac{\pi}{4}} r^3\cos\theta d\theta$$

$$= 2\times\frac{2}{3}\int_0^{\frac{\pi}{4}} a^3(\cos 2\theta)^{\frac{3}{2}}\cos\theta \, d\theta$$

$$= \frac{4}{3}a^3\int_0^{\frac{\pi}{4}} \left[1-2\sin^2\theta\right]^{\frac{3}{2}}\cos\theta \, d\theta$$

$Let \ \sqrt{2}\sin\theta = \sin\phi, \quad then \ \sqrt{2}\cos\theta d\theta = \cos\phi d\phi$



$$Dr = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 d\theta$$

$$= a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta \, d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$When\ \theta = 0,\quad 0 = \sin\phi,\quad i.e.\ \ \phi = 0$

$When\ \theta = \frac{\pi}{4},\quad 1 = \sin\phi,\quad i.e.\ \ \phi = \frac{\pi}{2}$

$$= \frac{4}{3\sqrt{2}} a^3 \int_0^{\frac{\pi}{2}} \left[1 - \sin^2\phi\right]^{\frac{3}{2}} \cos\phi\ d\phi$$

$$= \frac{4}{3\sqrt{2}} a^3 \int_0^{\frac{\pi}{2}} \cos^4\phi\ d\phi$$

$$= \frac{4}{3\sqrt{2}} a^3\ \frac{3.1}{4.2}.\frac{\pi}{2}$$

$$= \frac{\pi a^3}{4\sqrt{2}}$$

$$= 2a^2 \left[\frac{1}{2}\right]$$

$$= a^2$$

Therefore $\bar{x} = \dfrac{\frac{\pi a^3}{4\sqrt{2}}}{a^2} = \dfrac{\pi a}{4\sqrt{2}}$

$i.e.\quad (\bar{x}, \bar{y}) = \left(\dfrac{\pi a}{4\sqrt{2}}, 0\right)$

Centre of gravity of a solid revolution

Centre of gravity of a solid generated by the revolution of the area enclosed by the curve $y = f(x)$, the x axis and the ordinates $x = a$ and $x = b$, about the x-axis.	$\bar{x} = \dfrac{\int_a^b x\left(y^2 dx\right)}{\int_a^b y^2 dx}$	Since the CG of the solid of revolution lies on the x axis, $\bar{y} = 0$.

.

Centre of gravity of a solid generated by the revolution of the area enclosed by the curve $x = f(y)$, the y axis and the abscissae $y = a$ and $y = b$, about the y-axis.	$\bar{y} = \dfrac{\int_a^b y\left(x^2 dx\right)}{\int_a^b x^2 dx}$	Since the CG of the solid of revolution lies on the y axis, $\bar{x} = 0$.

.

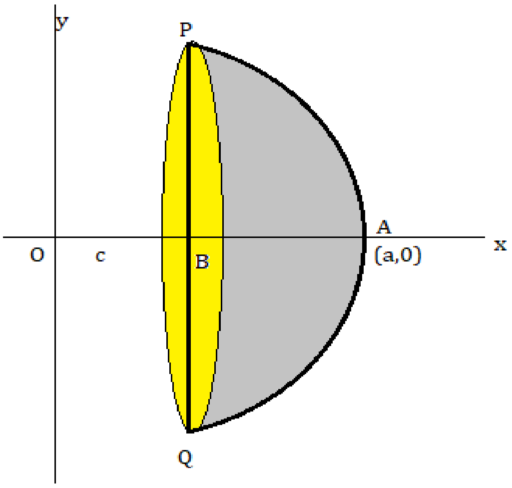
Centre of gravity of a solid generated by the revolution of the area enclosed by any two given curves about the x-axis, then where $a\ and\ b$ are the abscissae of the common points of intersection.	$\bar{x} = \dfrac{\int_a^b x\left(y_1^2 - y_2^2\right)\ dx}{\int_a^b \left(y_1^2 - y_2^2\right) dx}$	$\bar{y} = 0$

.

Centre of gravity of a solid generated by the revolution of the area enclosed by any two given curves about the y-axis, then where $a\ and\ b$ are the ordinates of the common points of intersection.	$\bar{y} = \dfrac{\int_a^b y\left(x_1^2 - x_2^2\right)\ dx}{\int_a^b \left(x_1^2 - x_2^2\right) dx}$	$\bar{x} = 0$

.

11 Find the centre of gravity of the segment of a sphere of radius a cut off by a plane at a distance c from the centre, and deduce the CG of a hemisphere.



The sphere is generated by rotating the circle $x^2+y^2=a^2$ about x-axis. By symmetry of the given solid, $\bar{y}=0$.

$$\bar{x}=\frac{\int_c^a x\left(y^2\right)dx}{\int_c^a y^2dx}$$

$$Nr=\int_c^a x\left(a^2-x^2\right)dx$$

$$=\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_c^a$$

$$=\left[\frac{a^2a^2}{2}-\frac{a^4}{4}-\frac{a^2c^2}{2}+\frac{c^4}{4}\right]$$

$$=\left[\frac{a^4}{2}-\frac{a^4}{4}-\frac{a^2c^2}{2}+\frac{c^4}{4}\right]$$

$$=\frac{1}{4}\left[a^4-2a^2c^2+c^4\right]$$

$$=\frac{1}{4}\left(a^2-c^2\right)^2$$

$$=\frac{1}{4}(a-c)^2(a+c)^2$$

$$Dr=\int_c^a\left(a^2-x^2\right)dx$$

$$=\left[a^2x-\frac{x^3}{3}\right]_c^a$$

$$=\left[a^3-\frac{a^3}{3}-a^2c+\frac{c^3}{3}\right]$$

$$=\left[\frac{2a^3}{3}-a^2c+\frac{c^3}{3}\right]$$

$$=\frac{1}{3}\left[2a^3-3a^2c+c^3\right]$$

$$=\frac{1}{3}(2a+c)\left(a^2-2ac+c^2\right)$$

$$=\frac{1}{3}(2a+c)(a-c)^2$$

Therefore $\bar{x}=\dfrac{\frac{1}{4}(a-c)^2(a+c)^2}{\frac{1}{3}(a-c)^2(2a+c)}=\dfrac{3}{4}\dfrac{(a+c)^2}{(2a+c)}$

In case the segment is a hemisphere, $c=0$. Therefore $\bar{x}=\dfrac{3a}{8}$, $\quad\bar{y}=0$

.

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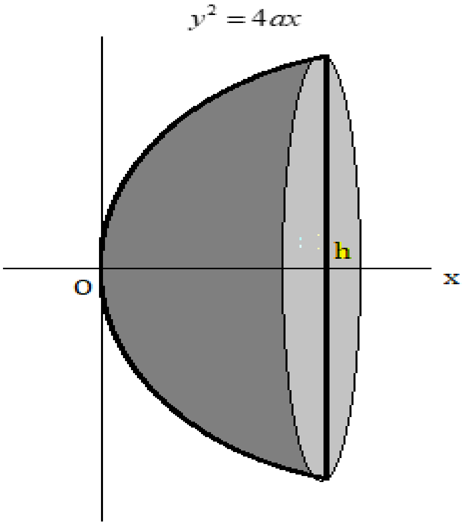
12 Find the centre of gravity of the volume formed by the revolution of the portion of the parabola $y^2 = 4ax$ cut off by the ordinate $x = h$ about the x-axis.



By symmetry of the given solid, $\overline{y} = 0$.

$$\overline{x} = \frac{\int_0^h x\left(y^2\right)dx}{\int_0^h y^2 dx}$$

$$= \frac{\int_0^h x\left(4ax\right)dx}{\int_0^h 4ax\ dx}$$

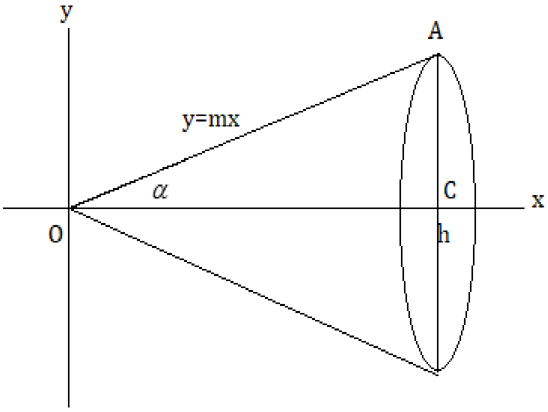
$$= \frac{4a\int_0^h x^2\ dx}{4a\int_0^h x\ dx}$$

$$\overline{x} = \frac{4a\left[\frac{x^3}{3}\right]_0^h}{4a\left[\frac{x^2}{2}\right]_0^h}$$

$$= \frac{\left[\frac{h^3}{3}\right]}{\left[\frac{h^2}{2}\right]}$$

$$= \frac{2h}{3}$$

13 Find the centre of gravity of a solid right circular cone of height h.



Let α be the semi vertical angle of the cone. The cone is generated by revolving the right triangle OAC about x axis. OA being the line $y = mx$, where $m = \tan\alpha$.

By symmetry of the given solid, $\overline{y} = 0$.

$$\overline{x} = \frac{\int_0^h x\left(y^2\right)dx}{\int_0^h y^2 dx}$$

$$\overline{x} = \frac{\int_0^h x\left(m^2x^2\right)dx}{\int_0^h m^2x^2dx}$$

$$\overline{x} = \frac{\int_0^h x^3dx}{\int_0^h x^2dx}$$

$$\overline{x} = \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h}$$

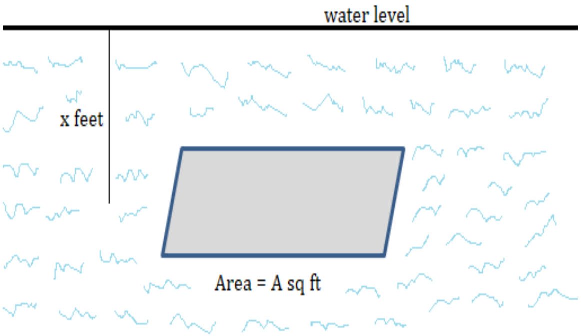
$$\overline{x} = \frac{3}{4}\frac{h^4}{h^3} = \frac{3h}{4}$$

.

Hydrostatic Force

A particle under water experiences pressure due to the weight of the water above. Consider a horizontal plate of area A sq. ft. at a depth of x feet below. The water directly above the plate exerts a **constant** force F equal to its weight on the plate.

This force is called hydrostatic force.



Suppose a **vertical plate** is submerged in water and we want to know the force that is exerted on the plate due to the pressure of the water. Hence the hydrostatic force on a vertical plate is not constant, since the pressure will vary with depth.

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The hydrostatic pressure at x meters below the water level is given by, $P = \rho g x$ where

ρ = the density of the water = 1000 kg/m^3

g = the gravitational acceleration = 9.81 m/s^2 .

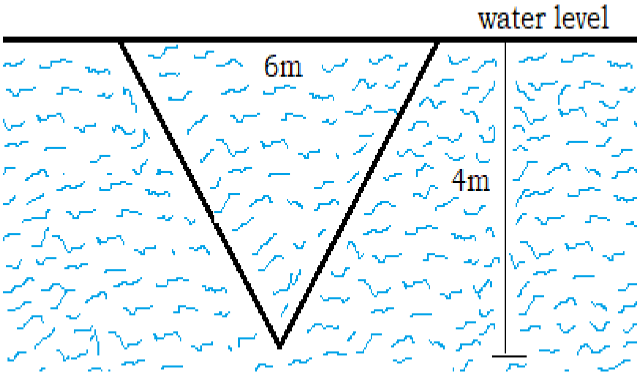
x = the distance between water level to a point at which pressure is measured

Suppose that a constant pressure P is acting on a surface with area A. Then the hydrostatic force that acts on the area is, F = PA

Note: All measurements should be in meters or convert into meters. If water is replaced by some other fluid, then the number 1000 kg/m^3 must be replaced by the weight of that fluid.

Solved Problems

1 Determine the hydrostatic force on the following isosceles triangular plate that is submerged vertically in water as shown below:



First let us fix the axis system. Let the water level be $y-$axis and positive $x-$axis towards the depth of the water. Hence $x = 4$ corresponds to the depth of the tip of the triangle.

Consider a strip of width Δx and length $2a$ in the plate. We use the property of similar triangles to determine a.

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$$\frac{3}{4}=\frac{a}{4-x}$$

$$a=\frac{3}{4}(4-x)$$

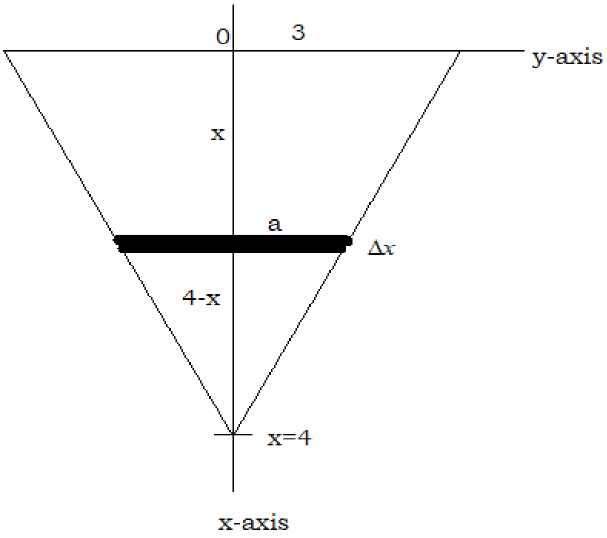
$$a=3-\frac{3}{4}x$$

since the pressure on this strip is constant, the pressure is given by

$$P=\rho gx$$

$$=1000\times 9.81\times x$$

$$=9810x$$



The area of each strip is $2a\Delta x$

and the hydrostatic force on each strip is, $F=PA=(9810x)(2a\Delta x)=19620x\left(3-\frac{3}{4}x\right)\Delta x$

Hence the hydrostatic force of the plate is $F=\sum_{i=1}^{n} 19620x_i\left(3-\frac{3}{4}x_i\right)\Delta x_i$

Taking the limit, we have

$$F=\underset{n\to\infty}{Lt}\sum_{i=1}^{n} 19620x_i\left(3-\frac{3}{4}x_i\right)\Delta x_i$$

$$=\int_0^4 19620x\left(3-\frac{3}{4}x\right)dx$$

$$=19620\int_0^4\left(3x-\frac{3}{4}x^2\right)dx$$

$$=19620\left[\frac{3x^2}{2}-\frac{x^3}{4}\right]_0^4$$

$$=19620\left[\frac{48}{2}-\frac{64}{4}\right]$$

$$=156960N$$

2 Find the hydrostatic force on a circular plate of radius 2m that is submerged 6 meters in the water.

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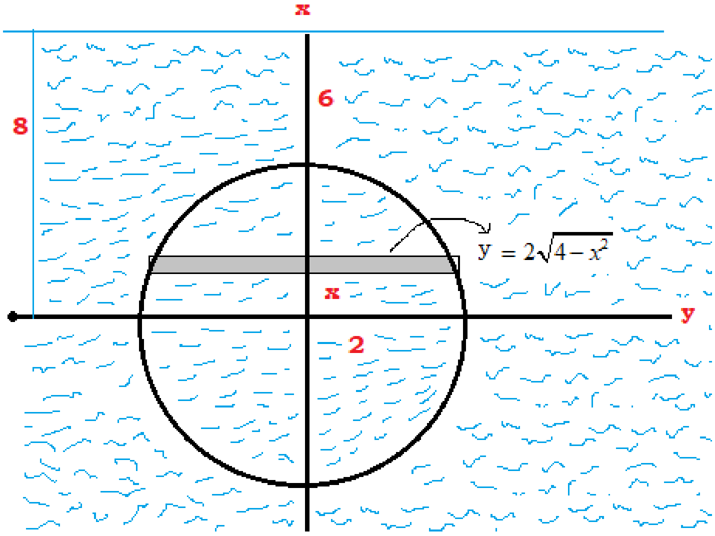
First let us fix the axis system. Let the origin of the axis is at the centre of the plate. Hence the water level is parallel to $y-$axis . Therefore the plate is at 6m depth and the centre is at 8m depth.



Consider a rectangular strip of width Δx and its length is $2\sqrt{4-x^2}$

since the pressure on this strip is constant, the pressure is given by

$$\begin{aligned}P &= \rho g x \\ &= 1000\times 9.81\times(8-x) \\ &= 9810(8-x)\end{aligned}$$

The area of each strip is $2\sqrt{4-x^2}.\Delta x$

and the hydrostatic force on each strip is,

$$F = PA = (9810)(8-x)\left(2\sqrt{4-x^2}.\Delta x\right) = 19620(8-x)\left(\sqrt{4-x^2}\right)\Delta x$$

Hence the hydrostatic force of the plate is $F = \sum_{i=1}^{n} 19620(8-x_i)\left(\sqrt{4-x_i^{\;2}}\right)\Delta x_i$

Taking the limit, we have

$$F = \underset{n\to\infty}{Lt} \sum_{i=1}^{n} 19620(8-x_i)\left(\sqrt{4-x_i^{\;2}}\right)\Delta x_i$$

$$F = \int_{-2}^{2} 19620(8-x)\left(\sqrt{4-x^2}\right) dx$$

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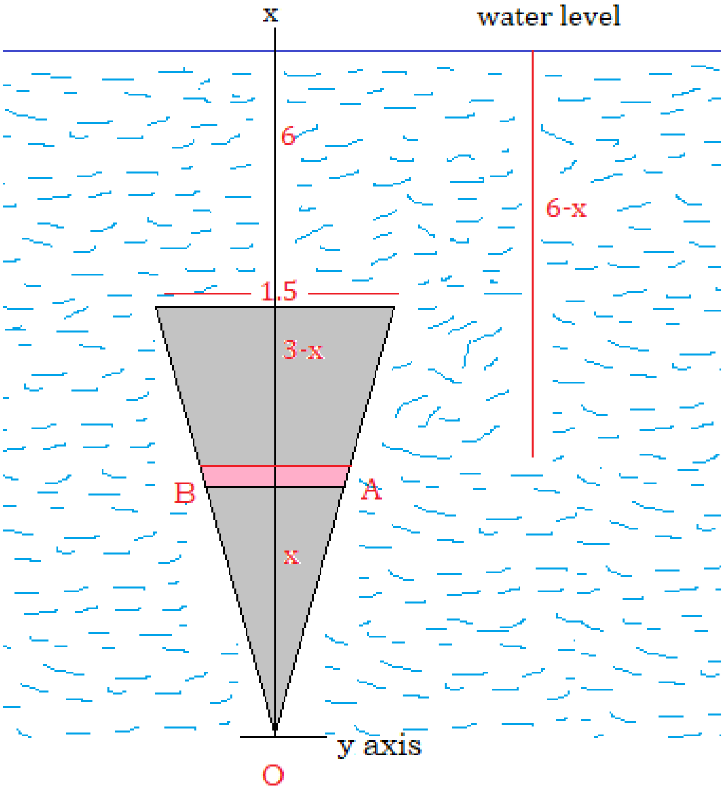
$$F = 19620 \int_{-2}^{2} 8\sqrt{4-x^2}\ dx - 19620 \int_{-2}^{2} x\sqrt{4-x^2}\ dx$$

$$F = 19620 \times 8 \times 2 \int_{0}^{2} \sqrt{4-x^2}\ dx - 19620\ (0),\ \{\text{By odd and even integral property}\}$$

$$F = 313920 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$
$$= 313920\ \left[2\sin^{-1}1\right]$$
$$= 313920\pi$$

3 An isosceles triangular plate 3m tall and 1.5m wide is located on a vertical wall of a swimming pool, the bottom vertex 6m below water level. Find the hydrostatic force F on the plate.

For our convenience we place the origin at the bottom vertex of the plate. Then the plate extends from $x=0$ to $x=3$. Therefore the depth at a given x is $6-x$.



Consider a strip AB of length w. Half of the length is $\frac{w}{2}$. From the properties of isosceles triangles, we have

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$$\frac{\frac{w}{2}}{x} = \frac{\frac{1.5}{2}}{3}$$

$$\frac{w}{2x} = \frac{1}{4}$$

$$w = \frac{x}{2}$$

Consider a strip of width Δx and its length is x as explained above.

since the pressure on this strip is constant, the pressure is given by

$$\begin{aligned} P &= \rho g x \\ &= 1000 \times 9.81 \times (6-x) \\ &= 9810(6-x) \end{aligned}$$

The area of each strip is $w.\Delta x = \frac{x}{2}.\Delta x$ and the hydrostatic force on each strip is, $F = PA = 9810(6-x)\left(\frac{x}{2}\Delta x\right) = 4905\left(6x - x^2\right)\Delta x$

Hence the hydrostatic force of the plate is $F = \sum_{i=1}^{n} 4905\left(6x_i - x_i^{\,2}\right)\Delta x_i$

Taking the limit, we have

$$F = \underset{n \to \infty}{Lt} \sum_{i=1}^{n} 4905\left(6x_i - x_i^{\,2}\right)\Delta x_i$$

$$\begin{aligned} F &= 4905\int_0^3 \left(6x - x^2\right)dx \\ &= 4905\left[3x^2 - \frac{x^3}{3}\right]_0^3 \\ &= 4905\left[27 - 9\right] \\ &= 88290N \end{aligned}$$

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4 Suppose the triangular plate is placed with the vertex at the top. Again consider the origin at the bottom. Hence for a given x the depth from the water level is $6-x$.

From the properties of triangles

$$\frac{\frac{w}{2}}{3-x}=\frac{\frac{1.5}{2}}{3}$$
$$\frac{w}{2(3-x)}=\frac{1}{4}$$
$$w=\frac{3-x}{2}$$

Hence the width $w(x)$ would be $\frac{3-x}{2}$.

Consider a strip of width Δx and its length is $\frac{3-x}{2}$ as explained above.

since the pressure on this strip is constant, the pressure is given by

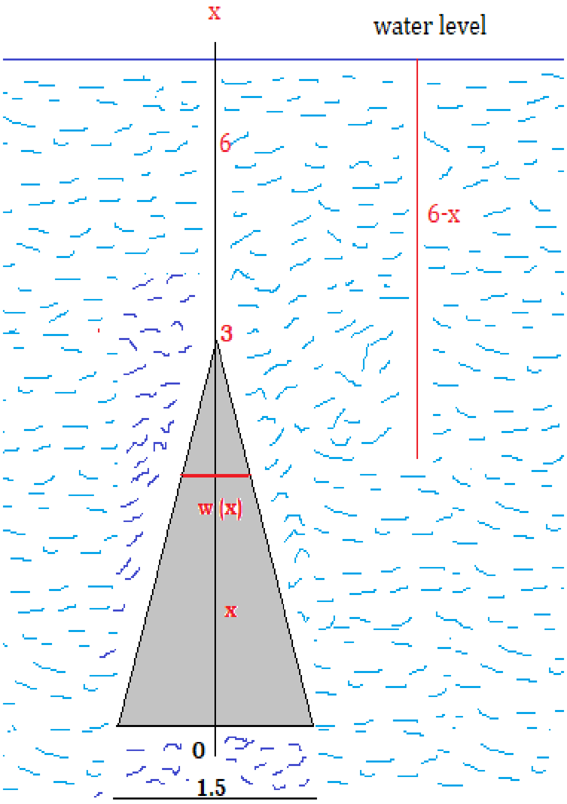
$$\begin{aligned}P&=\rho gx\\&=1000\times 9.81\times(6-x)\\&=9810(6-x)\end{aligned}$$

The area of each strip is $w.\Delta x=\frac{3-x}{2}.\Delta x$ and the hydrostatic force on each strip is,

$$\begin{aligned}F=PA&=9810(6-x)\left(\frac{3-x}{2}\Delta x\right)\\&=4905(6-x)(3-x)\Delta x\\&=4905\left(18-9x+x^2\right)\Delta x\end{aligned}$$

Hence the hydrostatic force of the plate is

$$F=\sum_{i=1}^{n}4905\left(18-9x_i+x_i^{\;2}\right)\Delta x_i$$



Taking the limit, we have the required force

$$F=\underset{n\to\infty}{Lt}\sum_{i=1}^{n}4905\left(18-9x_i+x_i^{\;2}\right)\Delta x_i$$

$$F=4905\int_0^3\left(18-9x+x^2\right)dx$$

$$=4905\left[18x-9\frac{x^2}{2}+\frac{x^3}{3}\right]_0^3$$

$$=4905\left[54-\frac{81}{2}+\frac{27}{3}\right]$$

$$=110362.5N$$

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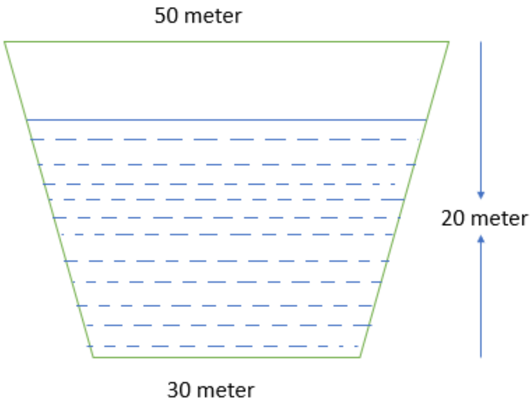
5. A dam has the shape as shown here. The height is 20m and the width is 50m at the top and 30m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4m from the top of the dam.

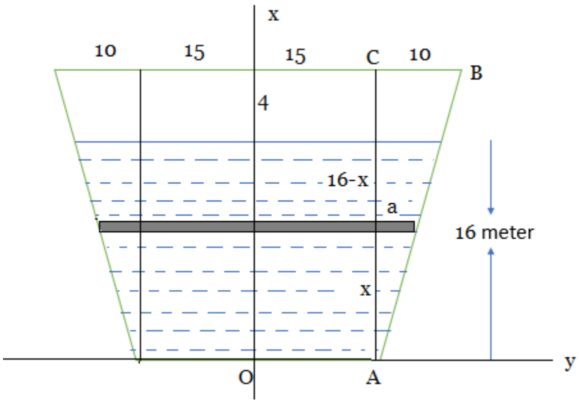


For our convenience we place the origin at the bottom vertex of the plate.

Then the water level of the dam extends from $x=0$ to $x=16$.

Therefore the depth at a given x is $16-x$.

Consider a strip of length a. From the properties of isosceles triangles, we have



$$\frac{a}{x}=\frac{10}{20}$$

$$a=\frac{x}{2}$$

Consider a strip of width Δx and its length is w as explained above. Therefore

$$w=2(a+15)$$

$$=2\left(\frac{x}{2}+15\right)$$

$$=x+30$$

since the pressure on this strip is constant, the pressure is given by

$$P=\rho g x$$

$$=1000\times 9.81\times(16-x)$$

$$=9810(16-x)$$

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The area of each strip is $w.\Delta x = (x+30).\Delta x$ and the hydrostatic force on each strip is, $F = PA = 9810(16-x)\left(x+30\right)\Delta x = 9810\left(480-14x-x^2\right)\Delta x$

Hence the hydrostatic force of the plate is $F = \sum_{i=1}^{n} 9810\left(480-14x_i - x_i^{\,2}\right)\Delta x_i$

Taking the limit, we have

$$F = \underset{n\to\infty}{Lt} \sum_{i=1}^{n} 9810\left(480-14x_i - x_i^{\,2}\right)\Delta x_i$$

$$F = 9810 \int_0^{16} \left(480-14x-x^2\right)dx$$

$$= 9810\left[480x - 7x^2 - \frac{x^3}{3}\right]_0^{16}$$

$$= 9810\left[7680-1792-1365\right]$$

$$= 44370630N$$

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UNIT V - MULTIPLE INTEGRALS

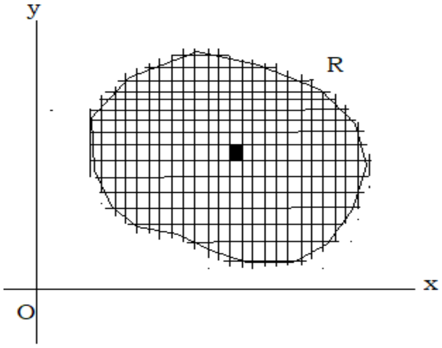
Introduction

The concept of multiple integrals is important in real life situations like evaluating the area of a region, volume of a solid where usual formulas are not useful. Also in Mechanics, it will be essential in evaluating mass, centre of gravity, moment of inertia of plane lamina and solid of volume V.

Double Integrals

We know that the $\int_a^b f(x)dx$ is defined as the limit of the sum $\sum_{i=1}^{n} \delta x_i f(x_i)$ as $n \to \infty$ where the range $b-a$ is divided into n parts and $x_1, x_2, \ldots, x_n$ are values of x lying in each interval δx_i. A double integral is its counterpart in two dimensions.

Let $f(x,y)$ be a single valued and bounded function of two variables in x, y defined in a closed region R. Divided the region into n sub-regions say, $\delta A_1, \delta A_2, \ldots, \delta A_n$. Let (x_i, y_i) be any point within the elementary area δA_i. When $n \to \infty$, the limit of sum $\sum_{i=1}^{n} \delta A_i f(x_i, y_i) = \iint_R f(x,y)dA$



If the region R is bounded by the curves $x = x_1,\ x = x_2,\ y = y_1,\ y = y_2$ then $\iint_R f(x,y)\, dx\, dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\, dx\, dy$

Evaluation of Double Integrals

(a) If x_1, x_2, y_1, y_2 are constants, then the order of integration is immaterial, provided the limits of integration are changed accordingly. Thus

$$\iint_R f(x,y)\, dx\, dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\, dy\, dx = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\, dx\, dy$$

(b) If x_1, x_2 are functions of y $\left(i.e.\ x_1 = \phi_1(y),\ \ x_2 = \phi_2(y)\right)$ and let y_1, y_2 are constants, then

$$\iint_R f(x,y)\, dx\, dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\, dx\, dy$$

(c) If y_1, y_2 are functions of x $\left(i.e.\ y_1 = \phi_1(x),\ \ y_2 = \phi_2(x)\right)$ and let x_1, x_2 are constants, then

$$\iint_R f(x,y)\, dx\, dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\, dy\, dx$$

From the above we observe that the integration is to be performed w.r.t. that variable having variable limits first and finally w.r.t the variable with constant limits.

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Rule: The limits of the inner integral are functions of x then the first integration is with respect to y and vice versa.

To evaluate $\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy$, we integrate $f(x,y)$ w.r.t x, treating y as a constant, getting a function of y (or constant), say $F(y)$ and then $F(y)$ is integrated w.r.t. y.

Note: If $f(x,y)=1$, then the double integral $\iint_A f(x,y)\,dx\,dy$ gives the area of A.

Triple Integrals

Let $f(x,y,z)$ be defined for all points in a finite region V of space. Let $\delta x\delta y\delta z$ be an elementary volume of the region V surrounding the points (x,y,z). Then

$\underset{\substack{\delta x\to 0\\ \delta y\to 0\\ \delta z\to 0}}{Lt}\sum\sum\sum f(x,y,z)\delta x\delta y\delta z$ is written as $\iiint_V f(x,y,z)dxdydz$ which is called the triple integral of $f(x,y,z)$ over the region V.

If the region V is bounded by the surfaces $x=x_1,\ \ x=x_2,\ \ y=y_1,\ \ y=y_2,\ \ z=z_1,\ \ z=z_2$ then

$$\iiint_V f(x,y,z)dxdydz = \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$$

Evaluation of Triple Integrals

(a) If $x_1,\, x_2,\, y_1,\, y_2,\, z_1,\, z_2$ are all constants, then the order of integration is immaterial, provided the limits of integration are changed accordingly. Thus

$$\iiint_V f(x,y,z)dxdydz = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx = \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$$

(b) If z_1, z_2 are functions of $x\ \&\ y\ \left(i.e.\ z_1=\phi_1(x,y),\ \ z_2=\phi_2(x,y)\right)$; y_1, y_2 are functions of $x\ \left(i.e.\ y_1=\psi_1(x),\ \ y_2=\psi_2(x)\right)$ while x_1, x_2 are constants, then integration is to be performed first w.r.t. z, then w.r.t. y and finally w.r.t. x. Thus

$$\iiint_V f(x,y,z)dxdydz = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx = \int_{x_1}^{x_2}\int_{\psi_1}^{\psi_2}\int_{\phi_1}^{\phi_2} f(x,y,z)\,dz\,dy\,dx$$

Note: If $f(x,y,z)=1$, then the triple integral $\iiint_V f(x,y,z)dxdydz$ gives the volume of the region V.

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Solved Problems

Double Integration in Cartesian coordinates

1 Evaluate $\int_0^1 \int_0^1 \frac{1}{\sqrt{\left(1-x^2\right)\left(1-y^2\right)}}\, dx\, dy$

Let $I = \int_0^1 \int_0^1 \frac{1}{\sqrt{\left(1-x^2\right)\left(1-y^2\right)}}\, dx\, dy$

$$= \int_0^1 \frac{1}{\sqrt{\left(1-y^2\right)}}\, dy \int_0^1 \frac{1}{\sqrt{\left(1-x^2\right)}}\, dx\, dy$$

$$= \left[\sin^{-1} x\right]_0^1 \left[\sin^{-1} y\right]_0^1$$

$$= \left[\sin^{-1} 1 - \sin^{-1} 0\right]\left[\sin^{-1} 1 - \sin^{-1} 0\right]$$

$$= \frac{\pi}{2} . \frac{\pi}{2}$$

2 Evaluate $\int_2^3 \int_1^2 \frac{1}{xy}\, dx\, dy$

Let $I = \int_2^3 \int_1^2 \frac{1}{xy}\, dx\, dy$

$$= \int_2^3 \frac{1}{y}\left[\log x\right]_1^2\, dy$$

$$= \int_2^3 \frac{1}{y}\left[\log 2 - \log 1\right] dy$$

$$= \log 2 \left[\log y\right]_2^3$$

= log 2 [log 3 – log 2]

= log2 . log 3/2

$\log 1 = 0, \quad \log a - \log b = \log \frac{a}{b}$

.

3 Evaluate: $\int_0^1 \int_1^2 x(x+y)\, dy\, dx$

Let I = $\int_0^1 \int_1^2 x(x+y)\, dy\, dx$

$$= \int_0^1 \left[x^2 y + x\frac{y^2}{2}\right]_1^2 dx$$

$$= \int_0^1 2x^2 + 2x - x^2 - \frac{x}{2}\, dx$$

$$= \int_0^1 x^2 + \frac{3x}{2}\, dx$$

$$= \left[\frac{x^3}{3} + \frac{3x^2}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{3}{4}$$

4 Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$

Let I = $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{\left(\sqrt{1+x^2}\right)^2 + y^2}\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1} 1 - \tan^{-1} 0\right] dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4}\left[\log\left(x + \sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\log\left(1+\sqrt{2}\right)\right]$$

5 Evaluate: $\int\limits_{0}^{5}\int\limits_{0}^{x^2} x(x^2+y^2)\,dy\,dx$

Let I = $\int\limits_{0}^{5}\int\limits_{0}^{x^2} x(x^2+y^2)\,dy\,dx$

$$=\int\limits_{0}^{5}\left[x^3y+\frac{xy^3}{3}\right]_0^{x^2} dx$$

$$=\int\limits_{0}^{5} x^5+\frac{x^7}{3}dx$$

$$=\left[\frac{x^6}{6}+\frac{x^8}{24}\right]_0^5$$

$$=\left[\frac{5^6}{6}+\frac{5^8}{24}\right]$$

6 Evaluate: $\int\limits_{0}^{1}\int\limits_{x}^{\sqrt{x}} xy(x+y)\,dx\,dy$

Let I = $\int\limits_{0}^{1}\int\limits_{x}^{\sqrt{x}} xy(x+y)\,dy\,dx$

$$=\int\limits_{0}^{1}\int\limits_{x}^{\sqrt{x}} x^2y+xy^2\;dy\;dx$$

$$=\int\limits_{0}^{1}\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_x^{\sqrt{x}} dx$$

$$=\int\limits_{0}^{1} \frac{x^3}{2}+\frac{x^{\frac{5}{2}}}{3}-\frac{x^4}{2}-\frac{x^4}{3}\;dx$$

$$=\left[\frac{x^4}{8}+\frac{2x^{\frac{7}{2}}}{21}-\frac{x^5}{6}\right]_0^1$$

$$=\frac{1}{8}+\frac{2}{21}-\frac{1}{6}=\frac{3}{56}$$

.

7 Evaluate $\int\limits_{0}^{\frac{\pi}{2}}\int\limits_{0}^{\frac{\pi}{2}} \sin(x+y)\,dx\,dy$

$$\int\limits_{0}^{\frac{\pi}{2}}\int\limits_{0}^{\frac{\pi}{2}} \sin(x+y)\,dx\,dy = \int\limits_{0}^{\frac{\pi}{2}}\left[\cos(x+y)\right]_0^{\frac{\pi}{2}}\,dy$$

$$=\int\limits_{0}^{\frac{\pi}{2}}\left[\cos\left(\frac{\pi}{2}+y\right)-\cos y\right]dy$$

$$=\int\limits_{0}^{\frac{\pi}{2}}\left[-\sin y-\cos y\right]dy$$

$$=\left[\cos y-\sin y\right]_0^{\frac{\pi}{2}}$$

$$=\left[(0-1)-(1-0)\right]=-2$$

8 Evaluate $\int\limits_{1}^{2}\int\limits_{0}^{y}\frac{1}{x^2+y^2}\,dx\,dy$

$$\int\limits_{1}^{2}\int\limits_{0}^{y}\frac{1}{x^2+y^2}\,dx\,dy=\int\limits_{1}^{2}\frac{1}{y}\left[\tan^{-1}\frac{x}{y}\right]_0^y dy$$

$$=\int\limits_{1}^{2}\frac{1}{y}\left(\tan^{-1}1-\tan^{-1}0\right)dy$$

$$=\int\limits_{1}^{2}\frac{1}{y}\left(\frac{\pi}{2}-0\right)dy$$

$$=\frac{\pi}{2}\left[\log y\right]_1^2$$

$$=\frac{\pi}{2}\log 2$$

.

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9 Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx = \int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{a^2}{2}\sin^{-1}1 - \frac{a^2}{2}\sin^{-1}0$$

$$= \frac{a^2}{2}\frac{\pi}{2}$$

10 Evaluate $\int_1^2 \int_1^3 xy^2\,dx\,dy$

$$\int_1^2 \int_1^3 xy^2\,dx\,dy = \int_1^2 y^2\left[\frac{x^2}{2}\right]_1^3 dy$$

$$= \int_1^2 y^2\left[\frac{9}{2}-\frac{1}{2}\right] dy$$

$$= 4\int_1^2 y^2\,dy$$

$$= 4\left[\frac{y^3}{3}\right]_1^2$$

$$= 4\left[\frac{8}{3}-\frac{1}{3}\right]$$

$$= 7$$

.

11 Evaluate $\iint_A xy\,dxdy$ **where** A **is the region bounded by the line** $x = 2a$ **and the curve** $x^2 = 4ay$ **and x-axis.**

$$\iint_A xy\,dxdy = \int_0^{2a}\int_0^{\frac{x^2}{4a}} xy\,dy\,dx$$

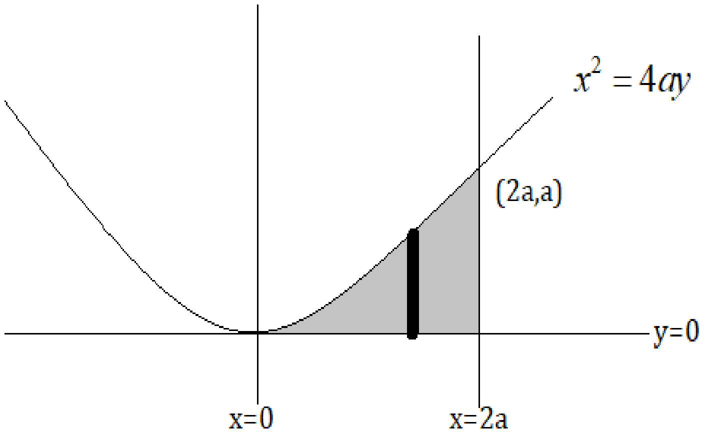
$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}} dx$$

$$= \int_0^{2a} \frac{x^5}{32a^2}\,dx$$

$$= \left[\frac{1}{32a^2}\frac{x^6}{6}\right]_0^{2a}$$

$$= \left[\frac{1}{32a^2}\times\frac{64a^6}{6}\right]$$

$$= \frac{a^4}{3}$$



In the figure x varies from x = 0 to x = 2a. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on y = 0 and upper end lies on $x^2 = 4ay$

i.e. $y = \frac{x^2}{4a}$

.

12 Evaluate $\iint_A xy\,dxdy$ where A is the region bounded by the first quadrant of the circle $x^2+y^2=a^2$.

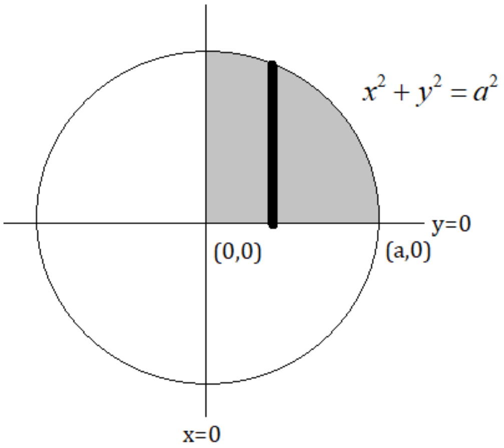
$$\iint_A xy\ dxdy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\ dy\ dx$$

$$= \frac{1}{2}\int_0^a x\left[y^2\right]_0^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\int_0^a x\left(a^2-x^2\right)dx$$

$$= \frac{1}{2}\left[a^2\frac{x^2}{2}-\frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]=\frac{a^4}{8}$$



In the figure x varies from x = 0 to x = a. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on y = 0 and upper end lies on $x^2+y^2=a^2$ i.e. $y^2=a^2-x^2$ *and* $y=\sqrt{a^2-x^2}$

Double integrals in polar coordinates

13 Evaluate: $\int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} r\ dr\ d\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} r\ dr\ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left[r^2\right]_0^{\sin\theta}\ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta\ d\theta$$

$$= \frac{1}{2}\frac{1}{2}\frac{\pi}{2}\ \text{\{by reduction formula\}}$$

$$= \frac{\pi}{8}$$

$$\int_0^{\frac{\pi}{2}}\sin^m\theta\,d\theta = \int_0^{\frac{\pi}{2}}\cos^m\theta\,d\theta = \frac{(m-1)(m-2)...}{(m)(m-2).....1}(1)\ or\ \left(\frac{\pi}{2}\right)$$

according as m is odd or even.

14 Evaluate: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2\ d\theta\ dr$

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2\ dr\ d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^3\theta\ d\theta$$

$$= 2\times\frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^3\theta\ d\theta\ \text{\{Even integral property\}}$$

$$= \frac{16}{3}\times\frac{2}{3}$$

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15 Evaluate: $\int\limits_{0}^{\frac{\pi}{2}}\int\limits_{1}^{1+\cos\theta} r\ dr\ d\theta$

$$\int\limits_{0}^{\frac{\pi}{2}}\int\limits_{1}^{1+\cos\theta} r\ dr\ d\theta = \int\limits_{0}^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{1}^{1+\cos\theta} d\theta$$

$$= \frac{1}{2}\int\limits_{0}^{\frac{\pi}{2}} (1+\cos\theta)^2 - 1\ d\theta$$

$$= \frac{1}{2}\int\limits_{0}^{\frac{\pi}{2}} 1+\cos^2\theta + 2\cos\theta - 1\ d\theta$$

$$= \frac{1}{2}\int\limits_{0}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} + 2\cos\theta\ d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\theta + \frac{1}{2\times 2}\sin 2\theta + 2\sin\theta\right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{1}{2}\frac{\pi}{2} + 2\right] = \frac{\pi}{8} + 1$$

16 Evaluate: $\int\limits_{0}^{\pi}\int\limits_{0}^{1-\cos\theta} r\ dr\ d\theta$

Let $I = \int\limits_{0}^{\pi}\int\limits_{0}^{1-\cos\theta} r\ dr\ d\theta$

$$= \int\limits_{0}^{\pi} \left[\frac{r^2}{2}\right]_{0}^{1-\cos\theta} d\theta$$

$$= \frac{1}{2}\int\limits_{0}^{\pi} (1-\cos\theta)^2\ d\theta$$

$$= \frac{1}{2}\int\limits_{0}^{\pi} 1+\cos^2\theta - 2\cos\theta\ d\theta$$

$$= \frac{1}{2}\left[\theta - 2\sin\theta\right]_{0}^{\pi} + \frac{1}{2}\int\limits_{0}^{\pi} \cos^2\theta\ d\theta$$

$$= \frac{1}{2}\left[\pi\right] + \frac{1}{2}\int\limits_{0}^{\pi} \frac{1+\cos 2\theta}{2}\ d\theta$$

$$= \frac{\pi}{2} + \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{0}^{\pi} = \frac{\pi}{2} + \frac{\pi}{4}$$

.

17 Evaluate: $\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int\limits_{0}^{2\cos\theta} d\theta\ dr$

Let $I = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int\limits_{0}^{2\cos\theta} dr\ d\theta$

$$= \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta\ d\theta$$

$$= 2\times 2\int\limits_{0}^{\frac{\pi}{2}} \cos\theta\ d\theta \quad \text{\{Even integral property\}}$$

$$= 4\left[\sin\theta\right]_{0}^{\frac{\pi}{2}}$$

$$= 4$$

18 Evaluate: $\int\limits_{0}^{\pi}\int\limits_{0}^{\cos\theta} r\ dr\ d\theta$

Let $I = \int\limits_{0}^{\pi}\int\limits_{0}^{\cos\theta} r\ dr\ d\theta$

$$= \int\limits_{0}^{\pi} \left[\frac{r^2}{2}\right]_{0}^{\cos\theta} d\theta$$

$$= \frac{1}{2}\int\limits_{0}^{\pi} \cos^2\theta\ d\theta$$

$$= \frac{1}{2\times 2}\int\limits_{0}^{\pi} 1+\cos 2\theta\ d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{0}^{\pi} = \frac{1}{4}\left[\pi\right]$$

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19 Evaluate: $\int\limits_0^{\pi}\int\limits_0^{\sin\theta} r\ dr\ d\theta$

Let $I = \int\limits_0^{\pi}\int\limits_0^{\sin\theta} r\ dr\ d\theta$

$$=\int\limits_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$$

$$=\frac{1}{2}\int\limits_0^{\pi}\sin^2\theta\ d\theta$$

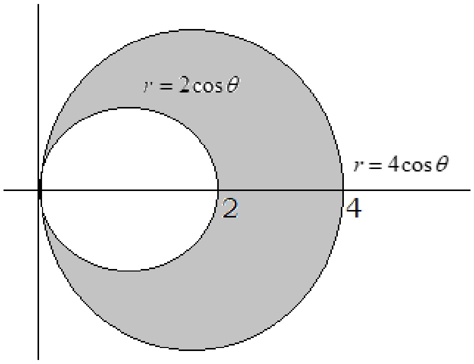
$$=\frac{1}{2\times 2}\int\limits_0^{\pi} 1-\cos 2\theta\ d\theta$$

$$=\frac{1}{4}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$=\frac{1}{4}[\pi]$$

.

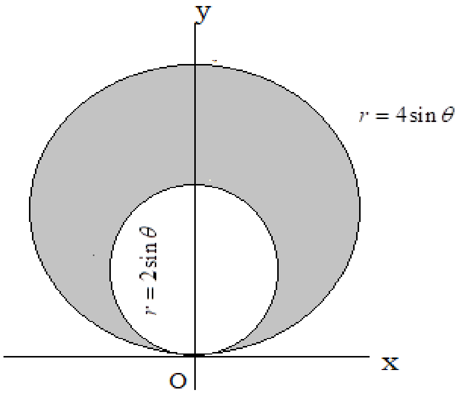
20 Evaluate $\iint r^3\ drd\theta$ **over the area bounded between the circles** $r = 2\cos\theta\ \ and\ \ r = 4\cos\theta$.



$$\iint r^3\ drd\theta = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int\limits_{2\cos\theta}^{4\cos\theta} r^3\ dr\ d\theta$$

$$=\frac{1}{4}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[r^4\right]_{2\cos\theta}^{4\cos\theta}\ d\theta$$

21 Evaluate $\iint r^3\ drd\theta$ **over the area bounded between the circles** $r = 2\sin\theta\ \ and\ \ r = 4\sin\theta$.



$$\iint r^3\ drd\theta = \int\limits_0^{\pi}\int\limits_{2\cos\theta}^{4\cos\theta} r^3\ dr\ d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[256\cos^4\theta - 16\cos^4\theta\right] \ d\theta$$

$$= \frac{240}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta \ d\theta$$

$$= \frac{240}{4}\times 2\int_{0}^{\frac{\pi}{2}} \cos^4\theta \ d\theta$$

$$= \frac{240}{4}\times 2 \ \frac{3.1}{4.2}\frac{\pi}{2}$$

$$= \frac{45}{2}\pi$$

$$= \frac{1}{4}\int_{0}^{\pi} \left[r^4\right]_{2\cos\theta}^{4\cos\theta} \ d\theta$$

$$= \frac{1}{4}\int_{0}^{\pi} \left[256\cos^4\theta - 16\cos^4\theta\right] \ d\theta$$

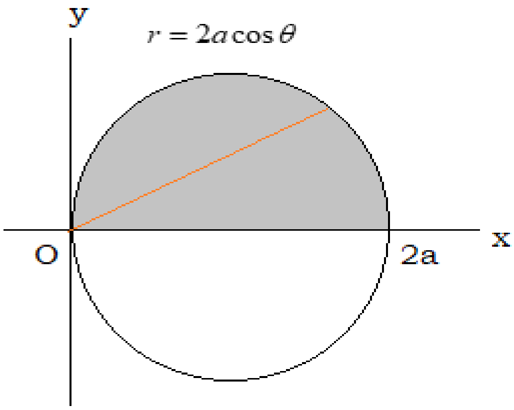
$$= \frac{240}{4}\int_{0}^{\pi} \cos^4\theta \ d\theta$$

$$= \frac{240}{4}\times 2\int_{0}^{\frac{\pi}{2}} \cos^4\theta \ d\theta$$

$$= \frac{240}{4}\times 2 \ \frac{3.1}{4.2}\frac{\pi}{2}$$

$$= \frac{45}{2}\pi$$

22 Evaluate $\iint r^2\sin\theta \ drd\theta$ over the area bounded between the semicircle $r = 2a\cos\theta$ above the initial line.

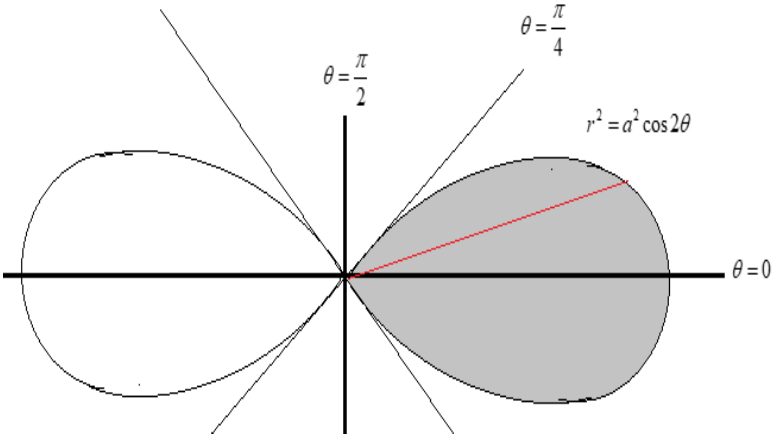


$$\iint r^2\sin\theta \ drd\theta = \int_{0}^{\frac{\pi}{2}}\int_{0}^{2a\cos\theta} r^2\sin\theta \ drd\theta$$

$$= \frac{1}{3}\int_{0}^{\frac{\pi}{2}} \left[r^3\right]_0^{2a\cos\theta} \sin\theta \ d\theta$$

$$= \frac{1}{3}\int_{0}^{\frac{\pi}{2}} 8a^3\cos^3\theta\sin\theta \ d\theta$$

23 Evaluate $\iint \frac{r}{\sqrt{a^2+r^2}} \ drd\theta$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$ above the initial line.



$$\iint \frac{r}{\sqrt{a^2+r^2}} \ drd\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}} \ drd\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\sqrt{a^2+r^2}\right]_0^{a\sqrt{\cos 2\theta}} \ d\theta$$

$$= \frac{8a^3}{3}\int_{0}^{\frac{\pi}{2}} \cos^3\theta \sin\theta \; d\theta$$

$$= \frac{8a^3}{3}\;\frac{2}{4.2}$$

$$= \frac{2a^3}{3}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\sqrt{a^2+a^2\cos 2\theta} - \sqrt{a^2}\right] \; d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[a\sqrt{1+\cos 2\theta} - a\right] \; d\theta$$

$$= a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\sqrt{2\cos^2\theta} - 1\right] \; d\theta$$

$$= 2a\int_{0}^{\frac{\pi}{4}} \sqrt{2}\cos\theta - 1 \;\; d\theta$$

$$= 2a\left[\sqrt{2}\sin\theta - \theta\right]_{0}^{\frac{\pi}{4}}$$

$$= 2a\left[\sqrt{2}\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]$$

$$= 2a\left[1-\frac{\pi}{4}\right]$$

Exercise

1 Evaluate the following integrals:

(i) $\displaystyle\int_{0}^{1}\int_{x}^{1} \frac{y}{x^2+y^2}\; dy\; dx$ (ii) $\displaystyle\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\; dy\; dx$

(iii) $\displaystyle\int_{0}^{1}\int_{0}^{1-x}\int^{1-x-y} xyz\; dz\; dy\; dx$ (iv) $\displaystyle\int_{0}^{\log 2}\int_{0}^{x}\int^{x+\log y} e^{x+y+z}\; dz\; dy\; dx$

2. Evaluate the following integrals:

(i) $\displaystyle\int_{0}^{\frac{\pi}{2}}\int_{x}^{4\sin\theta} r\,dr\; d\theta$ (ii) $\displaystyle\int_{0}^{2\pi}\int_{0}^{2} 4r - r^3\; dr\; d\theta$ (iii) $\displaystyle\int_{0}^{3\pi}\int_{\theta}^{2\theta} r^3\; dr\; d\theta$

3 Evaluate $\iint xy\, dxdy$ over the region in the positive quadrant bounded by the line $2x+3y=6$.

4 Evaluate $\iint x+y\;\, dxdy$ over the region in the positive quadrant of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

5 Evaluate $\iint x^2+y^2\;\, dxdy$ over the region bounded by the parabola $y^2=4x$ and its latus rectum.

6 Evaluate $\iint xy(x+y)\; dxdy$ over the area between the parabola $y^2=4x \; and \; y=x$.

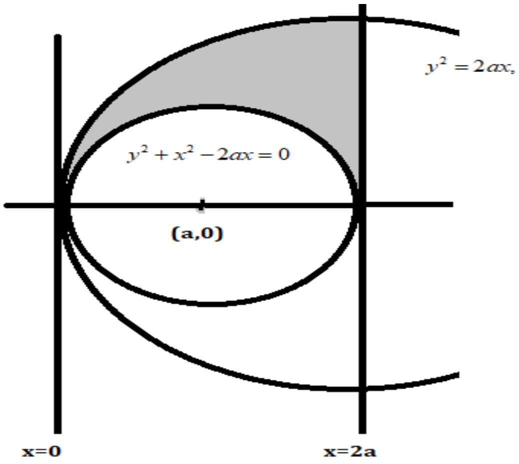
7 Evaluate $\iint x^2 \ dxdy$ over the region bounded by the hyperbola $xy = 4,\ y = 0,\ x = 1,\ and\ \ x = 2.$

Area Enclosed by Plane Curves

Area bounded by the lines $x = a,\ x = b,\ y = c,\ y = d\ (a < b\ \&\ c < d)$ is $\int_a^b \int_c^d dydx$. The limits of inner integration should be in variable, if necessary. In polar form the area is given by $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r\ drd\theta$

Sketch the region of integration for the double integral

$$\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\ dy\, dx$$



The region of integration is bounded by the lines $x = 0,\ \ x = 2a$ and the parabola and circle

$y = \sqrt{2ax}\ \ i.e.\ \ y^2 = 2ax,\ \ y = \sqrt{2ax - x^2}\ \ \ i.e.\ \ y^2 + x^2 - 2ax = 0$ respectively.

1 Find the area of the circle $x^2 + y^2 = a^2$

Area = 4 (area in first quadrant)

$$= 4\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\ dx$$

$$= 4\int_0^a \sqrt{a^2 - x^2}\ dx$$

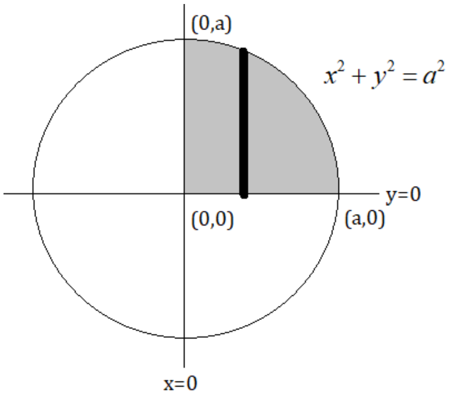
$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\frac{a^2}{2}\sin^{-1}1 - \frac{a^2}{2}\sin^{-1}0\right]$$

$$= 4\left[\frac{a^2}{2}\frac{\pi}{2}\right]$$

$$= \pi a^2$$

.



In the figure x varies from x = 0 to x = a. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on y = 0 and upper end lies on $x^2 + y^2 = a^2$ i.e. $y^2 = a^2 - x^2\ \ and\ \ y = \sqrt{a^2 - x^2}$.

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2 Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Area = 4 (area in first quadrant)

$$= 4\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dx\, dy$$

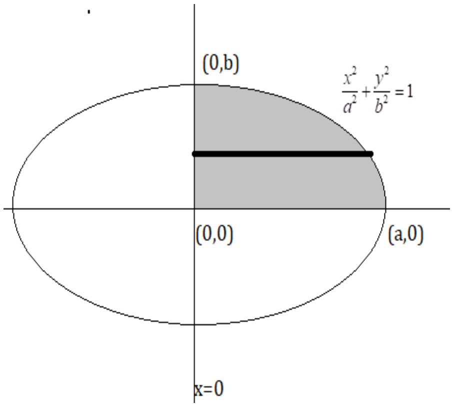
$$= 4\frac{a}{b}\int_0^b \sqrt{b^2-y^2}\; dy$$

$$= 4\frac{a}{b}\int_0^b \sqrt{b^2-y^2}\; dy$$

$$= 4\frac{a}{b}\left[\frac{y}{2}\sqrt{b^2-y^2}+\frac{b^2}{2}\sin^{-1}\frac{y}{b}\right]_0^b$$

$$= \frac{4a}{b}\left[\frac{b^2}{2}\sin^{-1}1-\frac{b^2}{2}\sin^{-1}0\right]$$

$$= \frac{4a}{b}\left[\frac{b^2}{2}\frac{\pi}{2}\right] = \pi ab$$



In the figure y varies from y = 0 to y = b. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on x = 0 and right end lies on

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad i.e. \quad \frac{x^2}{a^2}=1-\frac{y^2}{b^2}$$

$$x^2=a^2\left(\frac{b^2-y^2}{b^2}\right) \quad i.e. \quad x=\pm\frac{a}{b}\sqrt{b^2-y^2}$$

Note

$$\sin 0=0.\ \therefore \sin^{-1}0=0,\ \sin\frac{\pi}{2}=1.\ \therefore \sin^{-1}1=\frac{\pi}{2}$$

3 Find the area of the region bounded by the line $x=\frac{1}{4}$ and the parabola $y^2=4x$.

Area = 2 (area in the first quadrant)

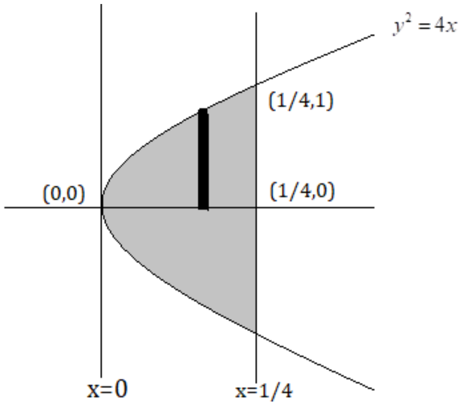
$$= 2\int_0^{\frac{1}{4}}\int_0^{2\sqrt{x}} dy\, dx$$

$$= 2\int_0^{\frac{1}{4}} 2\sqrt{x}\; dx$$

$$= 4\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{4}}$$

$$= 4\times\frac{2}{3}\times\left(\frac{1}{4}\right)^{\frac{3}{2}}$$

$$= \frac{1}{3}$$



In the figure x varies from x = 0 to x = 1/4. To find the limit for y, we take a strip parallel to the y – axis, it's lower end and upper end lies on $y^2=4x$ i.e. $y=\sqrt{4x}=\pm 2\sqrt{x}$

In the first quadrant, $y=0\ to\ y=2\sqrt{x}$

Note:

$$4\times\frac{2}{3}\times\left(\frac{1}{4}\right)^{\frac{3}{2}}=\frac{8}{3}\times\left(\frac{1}{4}\right)\times\left(\frac{1}{4}\right)^{\frac{1}{2}}=\frac{2}{3}\times\frac{1}{2}$$

.

4 Find the area between the parabolas

$x^2 = 4y \ \& \ y^2 = 4x$

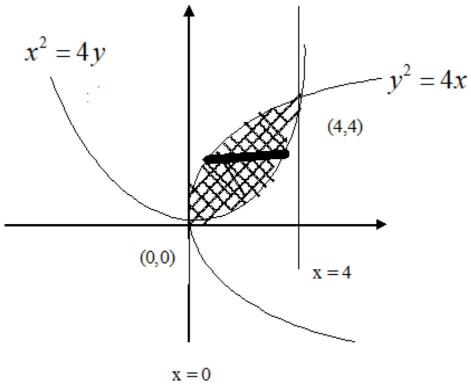
Area $= \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} dx\, dy$

$= \int_0^4 2\sqrt{y} - \frac{y^2}{4}\, dy$

$= \left[2\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{12} \right]_0^4$

$= 2.\frac{2}{3}.4^{\frac{3}{2}} - \frac{4^3}{12}$

$= \frac{16}{3}$



In the figure y varies from y = 0 to y = 4. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on

$y^2 = 4x$ *i.e.* $x = \frac{y^2}{4}$ and right end lies on

$x^2 = 4y \ \& \ x = 2\sqrt{y}$.

Note:

$2.\frac{2}{3}.4^{\frac{3}{2}} - \frac{4^3}{12} = \frac{4}{3}.4.\sqrt{4} - \frac{64}{12} = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$

.

5 Find the area between the parabolas

$5x^2 = 9y \ \& \ 3y^2 = 25x$

Area $= \int_0^5 \int_{\frac{3y^2}{25}}^{\frac{3}{\sqrt{5}}\sqrt{y}} dx\, dy$

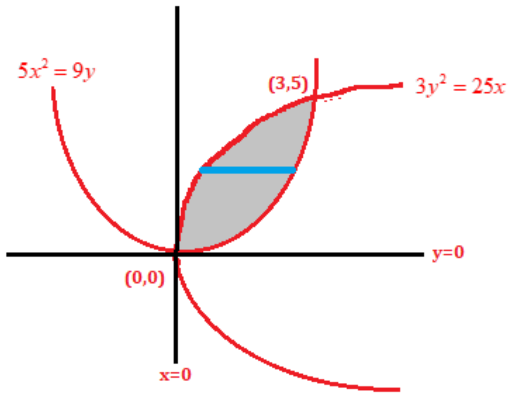
$= \int_0^5 \frac{3}{\sqrt{5}}\sqrt{y} - \frac{3y^2}{25}\, dy$

$= \left[\frac{3}{\sqrt{5}} \frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{25} \right]_0^5$

$= \frac{2}{\sqrt{5}}.5^{\frac{3}{2}} - \frac{5^3}{25}$

$= \frac{2}{\sqrt{5}}.5\sqrt{5} - 5$

$= 5$



In the figure y varies from y = 0 to y = 5. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on

$3y^2 = 25x$ *i.e.* $x = \frac{3y^2}{25}$ and right end lies on

$5x^2 = 9y \ \& \ x = \frac{3}{\sqrt{5}}\sqrt{y}$.

Note:

$2.\frac{2}{3}.4^{\frac{3}{2}} - \frac{4^3}{12} = \frac{4}{3}.4.\sqrt{4} - \frac{64}{12} = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$

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6 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

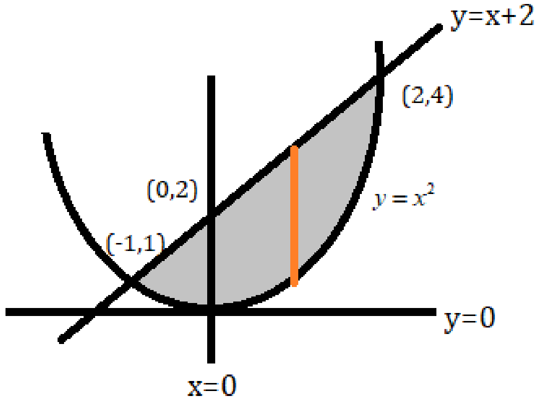
$$\text{Area} = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$$

$$= \int_{-1}^{2} x + 2 - x^2 \ dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[\left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$$

$$= \frac{9}{2}$$



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7 Find the smaller area between the line $x + y = a$ and the circle $x^2 + y^2 = a^2$.

$$\text{Area} = \int_{0}^{a}\int_{a-y}^{\sqrt{a^2-y^2}} dx\,dy$$

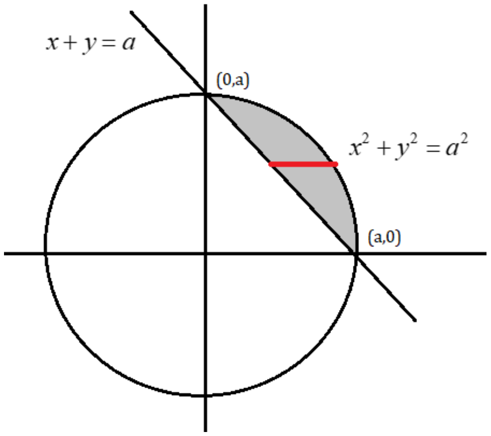
$$= \int_{0}^{a} \sqrt{a^2 - y^2} - (a - y)\ dy$$

$$= \left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a - \left[\frac{(a-y)^2}{-2}\right]_0^a$$

$$= \left[\frac{a^2}{2}\sin^{-1}1 - \frac{a^2}{2}\sin^{-1}0\right] - \left[0 + \frac{a^2}{2}\right]$$

$$= \frac{a^2}{2}.\frac{\pi}{2} - \frac{a^2}{2}$$

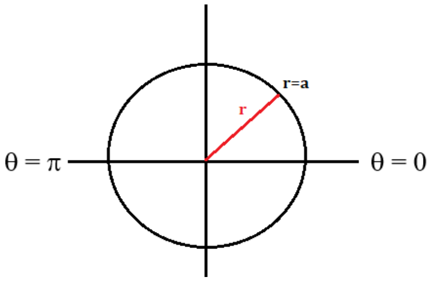
.



In the figure y varies from y = 0 to y = a. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $x + y = a; \quad x = a - y$ and right end lies on $x^2 + y^2 = a^2; \quad x = \sqrt{a^2 - y^2}$.

8 Find the area of the circle $r = a$.

In a circle r varies from $0 \text{ to } a$ and θ varies from $0 \text{ to } 2\pi$



Area = $\iint r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^a d\theta$$

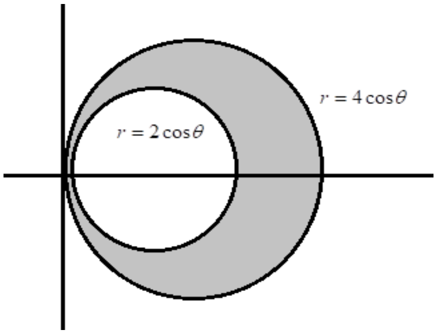
$$= \frac{a^2}{2}\int_0^{2\pi} d\theta$$

$$= \frac{a^2}{2}(2\pi - 0)$$

$$= \pi a^2$$

.

9 Find the area of the region outside the inner circle $r = 2\cos\theta$ and inside the outer circle $r = 4\cos\theta$ by double integration.



$r = 2\cos\theta$ is a circle with diameter 2. $r = 4\cos\theta$ is a circle with diameter 4. Initial line is the diameter.

Area = 2(area in the first quadrant)

$$= 2\iint r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\int_{2\cos\theta}^{4\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}} 16\cos^2\theta - 4\cos^2\theta\,d\theta$$

$$= 12\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 12\times\frac{1}{2}\times\frac{\pi}{2}$$

$$= 3\pi$$

10 Find the area of the cardioid $r = a(1+\cos\theta)$

From the figure

Area = 2 (area of upper part)

$$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi}(1+\cos\theta)^2\,d\theta$$

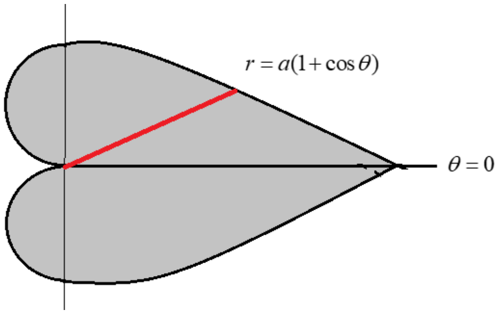
$$= a^2\int_0^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta$$

$$= 4a^2\times 2\int_0^{\frac{\pi}{2}}\cos^4\phi\,d\phi$$

$$= 8a^2\left[\frac{3.1}{4.2}.\frac{\pi}{2}\right]$$

$$= \frac{3}{2}\pi a^2$$



$2\cos^2\theta = 1+\cos 2\theta.\ \therefore 1+\cos\theta = 2\cos^2\frac{\theta}{2}$

Put $\frac{\theta}{2} = \phi$. Hence $\theta = 2\phi$ and $d\theta = 2\,d\phi$

when $\theta = 0,\quad \phi = 0$. *when* $\theta = \pi,\quad \phi = \frac{\pi}{2}$

Exercise

1 Find the area bounded by the parabola $y = x^2$ and the straight line $2x - y + 3 = 0$.

2 Find the area between the parabolas $y^2 = 4ax$ *and* $4by = x^2$.

3. Find the common area between the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2$.

4 Find the area of the region inside the cardioid $r = 1+\cos\theta$ and outside the circle $r = \frac{1}{2}$.

5. Find the area of the region bounded by the lemniscate $r^2 = 4\cos 2\theta$.

Change of Order of Integration

It sometimes happens that one iterated integral is either difficult or impossible to evaluate, whereas the other iterated integral can be evaluated easily. The change from one iterated integral to the other is called change of order of integration, since it involves changing from $dx\,dy$ *to* $dy\,dx$, or vice versa.

We know that the limits for inner integration are functions of variable, the change in the order of integration will result in changes in the limits of integration. i.e. the double integral $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$ will take the form $\int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx$ when the order of integration is changed.

To effect the change of order of integration, the region of integration is identified first and then new limits are fixed. (Constants limit for outer integral and variable limit for inner integration).

1. **Change the order of integration and evaluate**

$$\int_0^4 \int_y^4 \frac{x}{x^2+y^2}\,dy\,dx$$

Rewriting the given integral in proper order, we have

$$\int_0^4 \int_y^4 \frac{x}{x^2+y^2}\,dx\,dy$$

$\therefore$ the region of integration is bounded by

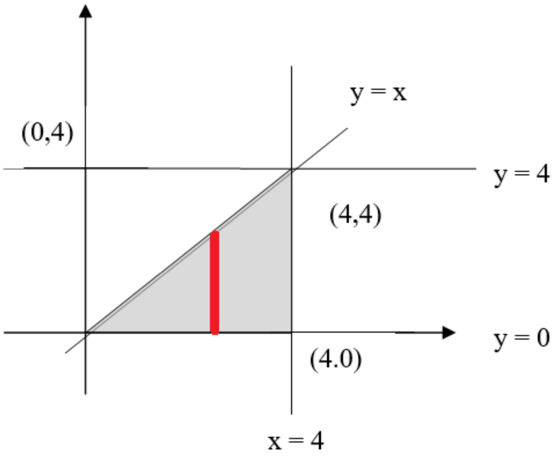
$x = y \qquad y = 0$

$x = 4 \qquad y = 4$

By changing the order, we have

$$\begin{aligned} I &= \int_0^4 \int_0^x \frac{x}{x^2+y^2}\,dy\,dx \\ &= \int_0^4 \left(\tan^{-1}\frac{y}{x} \right)_0^x dx \\ &= \int_0^4 \left(\tan^{-1}1 - \tan^{-1}0 \right) dx \\ &= \frac{\pi}{4} \int_0^4 dx \\ &= \frac{\pi}{4} \times 4 \end{aligned}$$

.



In the figure x varies from x = 0 to x = 4. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on y = 0 and upper end lies on $y = x$.

2. Change the order of integration and evaluate

$$\int_0^1\int_x^1 \frac{x}{x^2+y^2}\,dy\,dx$$

Given integral is in proper form.

By changing the order, we have

$$I = \int_0^1\int_0^y \frac{x}{x^2+y^2}\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 \left[\log\left(x^2+y^2\right)\right]_0^y\,dy$$

$$= \frac{1}{2}\int_0^1 \left[\log\left(2y^2\right)-\log\left(y^2\right)\right]dy$$

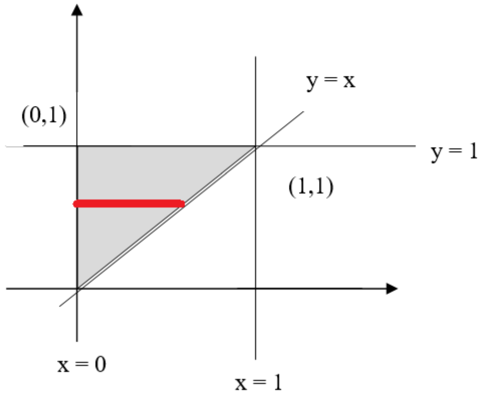
$$= \frac{1}{2}\int_0^1 \left[\log 2\right]dy$$

$$= \frac{1}{2}\left[\log 2\right]$$

$\therefore$ the region of integration is bounded by

$x=0 \qquad y=x$

$x=1 \qquad y=1$



In the figure y varies from y = 0 to y = 1. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on x = 0 and right end lies on $x=y$.

Note: $\log a-\log b=\log\dfrac{a}{b}$

$$\int_a^b k\;\;dy = k\left[y\right]_a^b = k(b-a) = k\;(UL-LL)$$

.

3. Change the order of integration and evaluate the integral $\int_0^a\int_x^a\left(x^2+y^2\right)dy\,dx$

Given integral is in proper form.

By changing the order, we have

$$I = \int_0^a\int_0^y\left(x^2+y^2\right)dx\,dy$$

$$= \int_0^a\left(\frac{x^3}{3}+y^2x\right)_0^y\;dy$$

$$= \int_0^a\left(\frac{y^3}{3}+y^3\right)dy$$

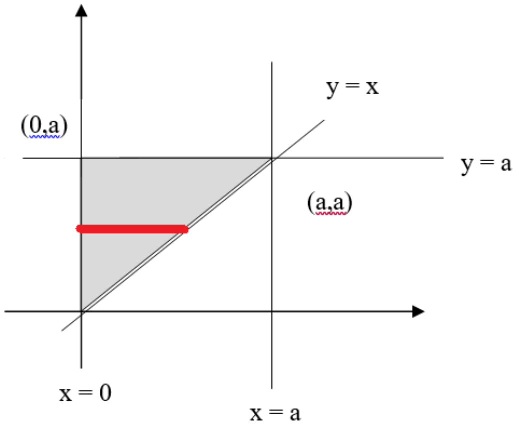
$$= \left(\frac{y^4}{12}+\frac{y^4}{4}\right)_0^a$$

$$= \left(\frac{a^4}{12}+\frac{a^4}{4}\right)$$

$\therefore$ the region of integration is bounded by

$x=0 \qquad y=x$

$x=a \qquad y=a$



In the figure y varies from y = 0 to y = a. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on x = 0 and right end lies on $x=y$.

.

4. Evaluate the integral $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} dy\,dx$ by changing the order of integration

Given integral is in proper form.
By changing the order, we have

$$I = \int_0^\infty \int_y^\infty x e^{-\frac{x^2}{y}} dx\,dy$$

$$I = \frac{1}{2}\int_0^\infty \int_{y^2}^\infty e^{-\frac{u}{y}} du\,dy \text{ \{Refer substitution\}}$$

$$= \frac{1}{2}\int_0^\infty -y\left[e^{-\frac{u}{y}}\right]_{y^2}^\infty dy$$

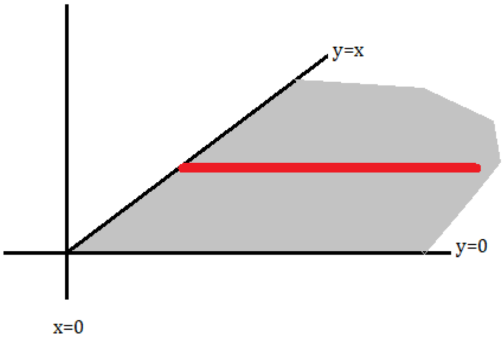
$$= \frac{1}{2}\int_0^\infty -y\left[0-e^{-y}\right] dy$$

$$= \frac{1}{2}\int_0^\infty y\,e^{-y}\,dy \quad = \quad \frac{1}{2}(1!)$$

$\therefore$ the region of integration is bounded by

$x=0 \qquad y=0$

$x=\infty \qquad y=x$



In the figure y varies from y = 0 to y = ∝. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $x=y$ and right end lies on $x=\infty$.

Let x² = u. Then 2x dx = du

when x = y, u = y². when x = ∝, u = ∝

$$\int_0^\infty e^{-x} x^{2-1}\,dx = \Gamma(2) = 1! \qquad \& \qquad e^{-\infty} = 0$$

.

5. Evaluate the integral $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy\,dx$ by changing the order of integration

Given integral is in proper form.

By changing the order, we have

$$I = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dx\,dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} (x)_0^y\,dy$$

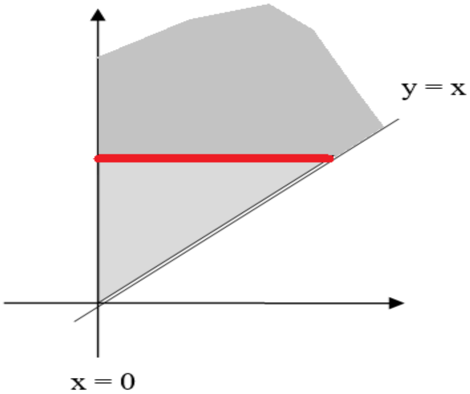
$$= \int_0^\infty e^{-y}\,dy$$

$$= \left(\frac{e^{-y}}{-1}\right)_0^\infty \quad = \quad 1$$

$\therefore$ the region of integration is bounded by

$x=0 \qquad y=x$

$x=\infty \qquad y=\infty$



In the figure y varies from y = 0 to y = ∝. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $x=0$ and right end lies on $x=y$.

6. Change the order of integration $\int_{0}^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\ dy\,dx$ **and evaluate it.**

Given integral is in proper form.

By changing the order, we have

$$I = \int_{0}^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy\ dx\,dy$$

$$= \int_{0}^{4a} y\left[\frac{x^2}{2}\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_{0}^{4a} y\left[4ay - \frac{y^4}{16a^2}\right] dy$$

$$= \frac{1}{2}\int_{0}^{4a} \left[4ay^2 - \frac{y^5}{16a^2}\right] dy$$

$$= \frac{1}{2}\left[4a\frac{y^3}{3} - \frac{y^6}{96a^2}\right]_{0}^{4a}$$

$$= \frac{1}{2}\left[4a\frac{64a^3}{3} - \frac{256\times 16a^6}{96a^2}\right]$$

$$= \frac{1}{2}\left[\frac{256a^4}{3} - \frac{256a^4}{6}\right]$$

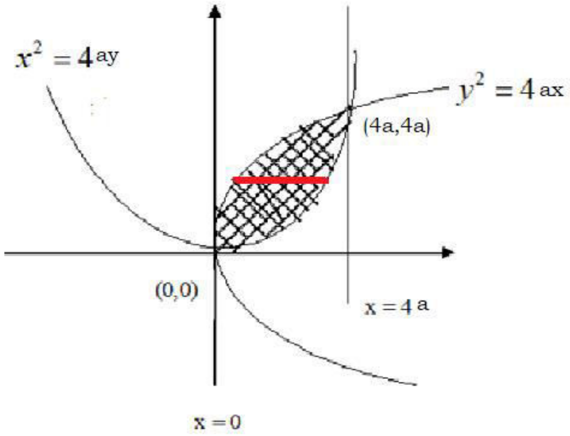
$$= \frac{256a^4}{2}\left[\frac{1}{3} - \frac{1}{6}\right]$$

$$= \frac{128a^4}{6}$$

$\therefore$ the region of integration is bounded by

$$x = 0 \qquad y = \frac{x^2}{4a} \qquad i.e.\ x^2 = 4ay$$

$$x = 4a \qquad y = 2\sqrt{ax} \qquad i.e.\ y^2 = 4ax$$



In the figure y varies from y = 0 to y = 4a. To find the limit for x, we take a strip parallel to the x – axis, it’s left end lies on $y^2 = 4ax;\ x = \frac{y^2}{4a}$ and right end lies on $x^2 = 4ay;\ x = 2\sqrt{ay}$.

7. Change the order of integration $\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\,dx$ **and evaluate it.**

Given integral is in proper form.

By changing the order, we have

$$I = \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} dx\,dy$$

$$= \int_0^4 2\sqrt{y} - \frac{y^2}{4}\,dy$$

$$= \int_0^4 2\,y^{\frac{1}{2}} - \frac{y^2}{4}\,dy$$

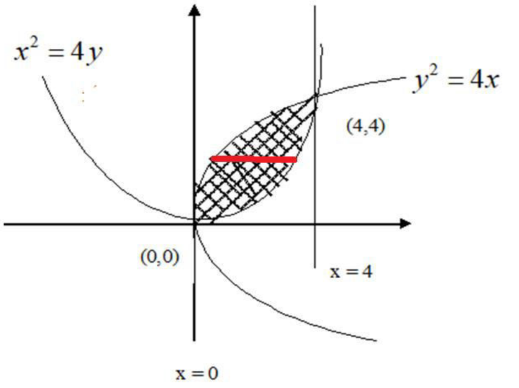
$$= \left[\frac{4}{3}y^{\frac{3}{2}} - \frac{y^3}{12}\right]_0^4$$

$$= \left[\frac{4}{3}4^{\frac{3}{2}} - \frac{64}{12}\right] = \frac{16}{3}$$

$\therefore$ the region of integration is bounded by

$x = 0 \qquad y = \frac{x^2}{4} \qquad i.e.\ x^2 = 4y$

$x = 4 \qquad y = 2\sqrt{x} \qquad i.e.\ y^2 = 4x$



In the figure y varies from y = 0 to y = 4. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $y^2 = 4x;\ x = \frac{y^2}{4}$ and right end lies on $x^2 = 4y;\ x = 2\sqrt{y}$.

.

8. Evaluate by changing its order $\int_0^1 \int_y^{\sqrt{y}} \frac{x}{x^2+y^2}\,dy\,dx$

Rewriting the given integral in proper order, we have

$$\int_0^1 \int_y^{\sqrt{y}} \frac{x}{x^2+y^2}\,dx\,dy$$

By changing the order, we have

$$I = \int_0^1 \int_{x^2}^{x} \frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_0^1 \left(\tan^{-1}\frac{y}{x}\right)_{x^2}^{x} dx$$

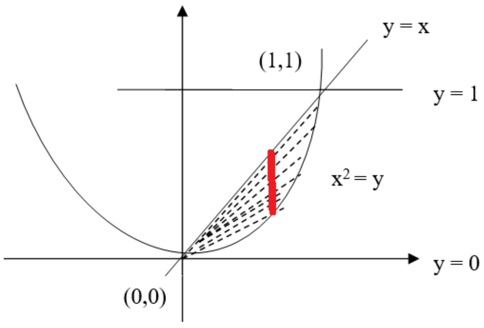
$$= \int_0^1 \frac{\pi}{4} - \tan^{-1} x\,dx \qquad \sin ce\ \ \tan^{-1} 1 = \frac{\pi}{4}$$

$$= \left[\frac{\pi}{4}x - \left\{x\tan^{-1} x - \frac{1}{2}\log\left(1+x^2\right)\right\}\right]_0^1 = \frac{1}{2}\log 2$$

$\therefore$ the region of integration is bounded by

$y = 0 \qquad x = y$

$y = 1 \qquad x = \sqrt{y} \quad i.e.\ x^2 = y$



In the figure x varies from x = 0 to x = 1. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on $y = x^2$ and upper end lies on $y = x$.

9. Change the order of integration in $\int_0^4 \int_0^{\frac{3}{4}\sqrt{16-x^2}} x\, dy\, dx$ **and hence evaluate.**

Given integral is in proper form.

By changing the order, we have

$$I = \int_0^3 \int_0^{\frac{4}{3}\sqrt{9-y^2}} x\, dx\, dy$$

$$= \frac{1}{2}\int_0^3 \left[x^2\right]_0^{\frac{4}{3}\sqrt{9-y^2}} dy$$

$$= \frac{1}{2}\frac{16}{9}\int_0^3 \left(9-y^2\right) dy$$

$$= \frac{8}{9}\left(9y - \frac{y^3}{3}\right)_0^3$$

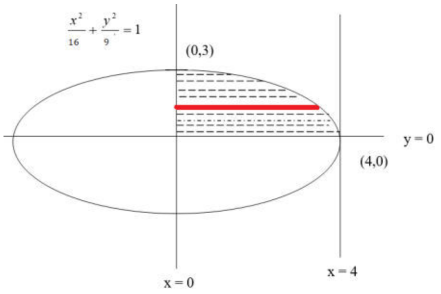
$$= \frac{8}{9}\left(27-9\right)$$

$\therefore$ the region of integration is bounded by

$$x=0,\ x=4, \quad y=0,\ y=\frac{3}{4}\sqrt{16-x^2}$$

$$y^2 = \frac{9}{16}\left(16-x^2\right) \qquad i.e.\ \ 16y^2 = 144-9x^2$$

$$i.e.\ \ 9x^2+16y^2=144 \quad i.e.\ \ \frac{x^2}{16}+\frac{y^2}{9}=1$$



In the figure y varies from y = 0 to y = 3. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $x=0$ and right end lies on

$$\frac{x^2}{16}+\frac{y^2}{9}=1 \qquad i.e.\ \ \frac{x^2}{16}=1-\frac{y^2}{9}$$

$$x^2 = 16\left(\frac{9-y^2}{9}\right) \quad i.e.\ \ x=\frac{4}{3}\sqrt{9-y^2}$$

10. Evaluate by changing the order of integration $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\, dy\, dx$

Rewriting the given integral in proper order, we have $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\, dx\, dy$

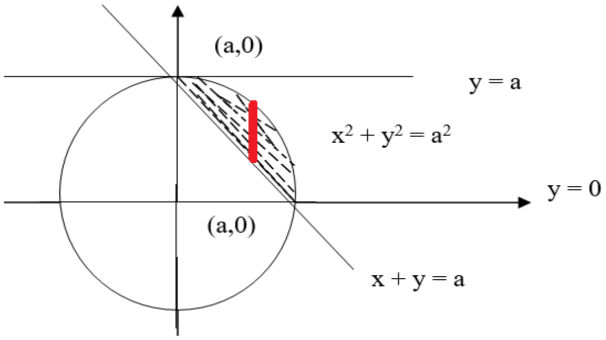
By changing the order, we have

$$I = \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\, dy\, dx$$

$\therefore$ the region of integration is bounded by

$y=0 \qquad x=a-y \quad i.e.\ \ x+y=a$

$y=a \qquad x=\sqrt{a^2-y^2} \quad i.e.\ \ x^2+y^2=a^2$



$$= \frac{1}{2}\int_0^a \left[y^2\right]_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left(a^2-x^2\right)-(a-x)^2\,dx$$

$$= \frac{1}{2}\left[a^2x-\frac{x^3}{3}-\frac{(a-x)^3}{-3}\right]_0^a$$

$$= \frac{1}{2}\left[a^3-\frac{a^3}{3}-\frac{a^3}{3}\right]$$

$$= \frac{a^3}{6}$$

In the figure x varies from x = 0 to x = a. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on $x+y=a;\ y=a-x$ and upper end lies on $x^2+y^2=a^2;\ y^2=a^2-x^2;\ y=\sqrt{a^2-x^2}$ **.**

.

11. Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} x+y\ dx\,dy$ **by change of order of integration.**

Solution: Given integral is in proper form.

By changing the order, we have

$$I = \int_1^2 \int_0^{4-x^2} x+y\ dy\ dx$$

$$= \int_1^2 \left[xy+\frac{y^2}{2}\right]_0^{4-x^2} dx$$

$$= \int_1^2 x\left(4-x^2\right)+\tfrac{1}{2}\left(4-x^2\right)^2\ dx$$

$$= \int_1^2 \frac{1}{2}x^4-x^3-4x^2+4x+8\ dx$$

$$= \left[\frac{x^5}{10}-\frac{x^4}{4}-\frac{4x^3}{3}+2x^2+8x\right]_1^2$$

$$= \left[\frac{32}{10}-4-\frac{32}{3}+8+16\right]-\left[\frac{1}{10}-\frac{1}{4}-\frac{4}{3}+2+8\right]$$

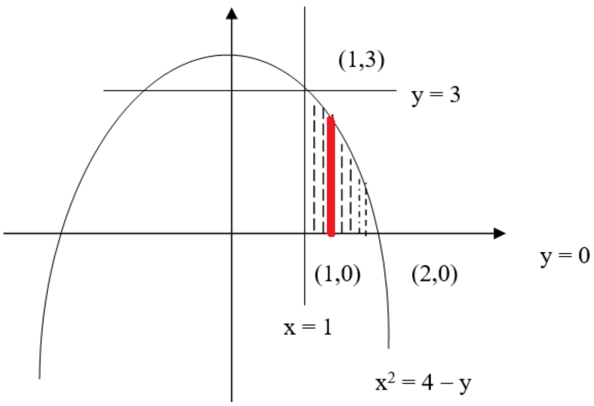
$$= \frac{241}{60}$$

.

$\therefore$ the region of integration is bounded by

$y=0 \qquad x=1$

$y=3 \qquad x=\sqrt{4-y} \qquad i.e.\ x^2=4-y$



In the figure x varies from x = 1 to x = 2. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on $y=0$ and upper end lies on $x^2=4-y;\quad y=4-x^2$ **.**

12. Change the order of integration and hence evaluate $\int_0^1 \int_{y^2}^{2-y} xy \; dx\,dy$

Given integral is in proper form.

By changing the order, we have

$$I = \int_0^1 \int_0^{\sqrt{x}} xy \; dy\,dx + \int_1^2 \int_0^{2-x} xy \; dy\,dx$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^{\sqrt{x}} dx + \int_1^2 x\left(\frac{y^2}{2}\right)_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^2\,dx + \frac{1}{2}\int_1^2 x(2-x)^2\,dx$$

$$= \frac{1}{2}\int_0^1 x^2\,dx + \frac{1}{2}\int_1^2 4x + x^3 - 4x^2\,dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3}\right)_0^1 + \frac{1}{2}\left(2x^2 + \frac{x^4}{4} - \frac{4x^3}{3}\right)_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8+4-\frac{32}{3}\right) - \left(2+\frac{1}{4}-\frac{4}{3}\right)\right]$$

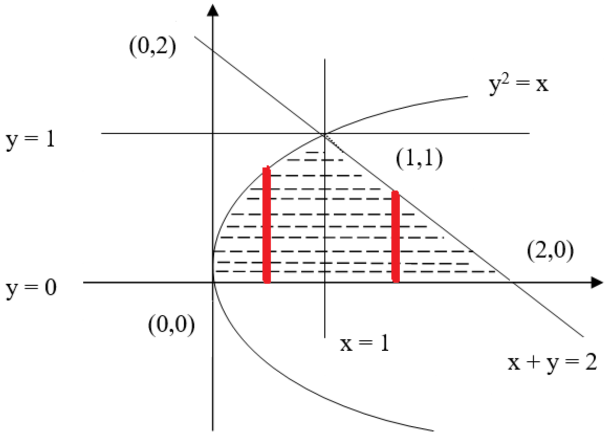
$$= -\frac{23}{12}$$

.

∴ the region of integration is bounded by

$y = 0 \qquad x = y^2$

$y = 1 \qquad x = 2 - y \qquad i.e.\ x + y = 2$



In the first region x varies from 0 *to* 1. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on $y = 0$ and upper end lies on $y^2 = x; \quad y = \sqrt{x}$.

In the second region x varies from 1 *to* 2. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on $y = 0$ and upper end lies on $x + y = 2; \quad y = 2 - x$.

https://doi.org/10.5281/zenodo.15288481

13. Change the order of integration and hence evaluate $\int_0^1 \int_y^{2-y} xy\ dx\,dy$

Given integral is in proper form.

By changing the order, we have

$$I = \int_0^1 \int_0^x xy\ dy\,dx + \int_1^2 \int_0^{2-x} xy\ dy\,dx$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^x dx + \int_1^2 x\left(\frac{y^2}{2}\right)_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^3\,dx + \frac{1}{2}\int_1^2 x(2-x)^2\,dx$$

$$= \frac{1}{2}\int_0^1 x^3\,dx + \frac{1}{2}\int_1^2 4x + x^3 - 4x^2\,dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4}\right)_0^1 + \frac{1}{2}\left(2x^2 + \frac{x^4}{4} - \frac{4x^3}{3}\right)_1^2$$

$$= \frac{1}{8} + \frac{1}{2}\left[\left(8 + 4 - \frac{32}{3}\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$

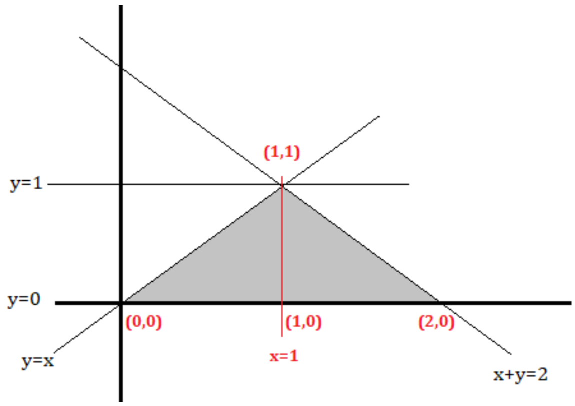
$$= \frac{1}{3}$$

..

$\therefore$ the region of integration is bounded by

$y = 0 \qquad x = y$

$y = 1 \qquad x = 2 - y \qquad i.e.\ x + y = 2$



In the first region x varies from $0\ to\ 1$. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on $y = 0$ and upper end lies on $y = x$.

In the second region x varies from $1\ to\ 2$. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on $y = 0$ and upper end lies on $x + y = 2;\quad y = 2 - x$.

14. Change the order of integration and hence evaluate $\int\limits_{0}^{a}\int\limits_{\frac{x^2}{a}}^{2a-x} xy\ dy\,dx$

Given integral is in proper form.

By changing the order, we have

$$I = \int\limits_{0}^{a}\int\limits_{0}^{\sqrt{ay}} xy\ dx\,dy + \int\limits_{1}^{2a}\int\limits_{0}^{2a-y} xy\ dx\,dy$$

$$= \int\limits_{0}^{a} y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy + \int\limits_{a}^{2a} y\left(\frac{x^2}{2}\right)_0^{2a-y} dy$$

$$= \frac{a}{2}\int\limits_{0}^{a} y^2\,dy + \frac{1}{2}\int\limits_{a}^{2a} y(2a-y)^2\,dy$$

$$= \frac{a}{2}\int\limits_{0}^{a} y^2\,dy + \frac{1}{2}\int\limits_{a}^{2a} 4a^2y + y^3 - 4ay^2\,dy$$

$$= \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a + \frac{1}{2}\left(2ay^2 + \frac{y^4}{4} - \frac{4ay^3}{3}\right)_a^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[\left(8a^4 + 4a^4 - \frac{32a^4}{3}\right) - \left(2a^4 + \frac{a^4}{4} - \frac{4a^4}{3}\right)\right]$$

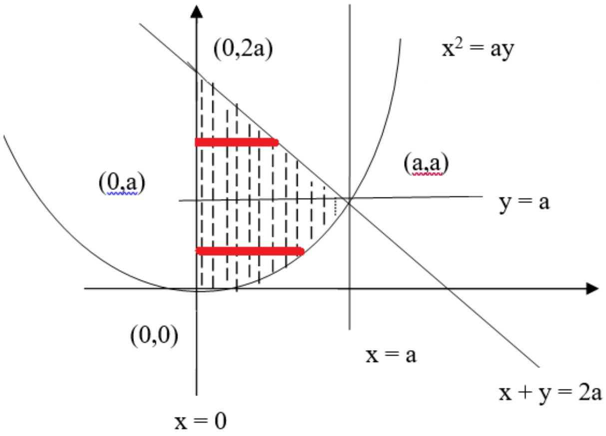
$$= \frac{9}{24}a^4$$

.

$\therefore$ the region of integration is bounded by

$x = 0 \qquad y = \frac{x^2}{a} \qquad i.e. \qquad x^2 = ay$

$x = a \qquad y = 2a - x \qquad i.e. \qquad x + y = 2a$



In the first region y varies from $0\ to\ a$. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $x = 0$ and right end lies on $x^2 = ay; \quad x = \sqrt{ay}$.

In the second region y varies from $a\ to\ 2a$. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $x = 0$ and right end lies on $x + y = 2a; \quad x = 2a - y$.

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15. Change the order of integration and hence evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$

Given integral is in proper form.

By changing the order, we have

$$I = \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

$$= \int_1^{\sqrt{2}} \left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy + \int_0^1 \left[\sqrt{x^2+y^2}\right]_0^y dy$$

$$= \int_1^{\sqrt{2}} \sqrt{2} - \sqrt{y^2}\ \ dy + \int_0^1 \sqrt{2y^2} - \sqrt{y^2}\ \ dy$$

$$= \int_1^{\sqrt{2}} \sqrt{2} - y\ \ dy + \int_0^1 \left(\sqrt{2}-1\right) y\ dy$$

$$= \left(\sqrt{2}\, y - \frac{y^2}{2}\right)_1^{\sqrt{2}} + \left(\sqrt{2}-1\right)\left(\frac{y^2}{2}\right)_0^1$$

$$= (2-1) - \left(\sqrt{2} - \frac{1}{2}\right) + \frac{1}{2}\left(\sqrt{2}-1\right)$$

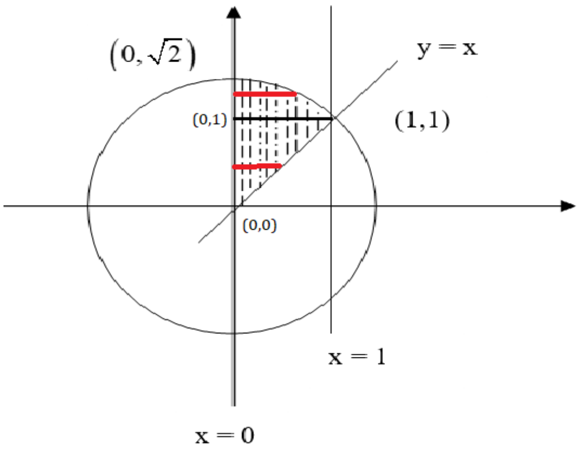
$$= \frac{1}{2}\left(2-\sqrt{2}\right)$$

.

$\therefore$ the region of integration is bounded by

$x = 0$ $\quad$ $y = x$

$x = 1$ $\quad$ $y = \sqrt{2-x^2}$ $\quad$ *i.e.* $\quad$ $x^2 + y^2 = 2$



In the bottom region y varies from 0 *to* 1. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $x = 0$ and right end lies on $x = y$.

In the top region y varies from 1 *to* $\sqrt{2}$. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $x = 0$ and right end lies on $x^2 + y^2 = 2;\quad x = \sqrt{2-y^2}$.

16. Evaluate $\int_0^3 \int_0^{\sqrt{4-y}} x+y \; dx\,dy$ by change the order of integration.

Given integral is in proper form.

By changing the order, we have

$$I = \int_0^1 \int_0^3 x+y \; dy\,dx + \int_1^2 \int_0^{4-x^2} x+y \; dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^3 dx \; + \; \frac{241}{60} \quad \text{\{Refer example 10\}}$$

$$= \int_0^1 3x + \frac{9}{2} \; dx \; + \; \frac{241}{60}$$

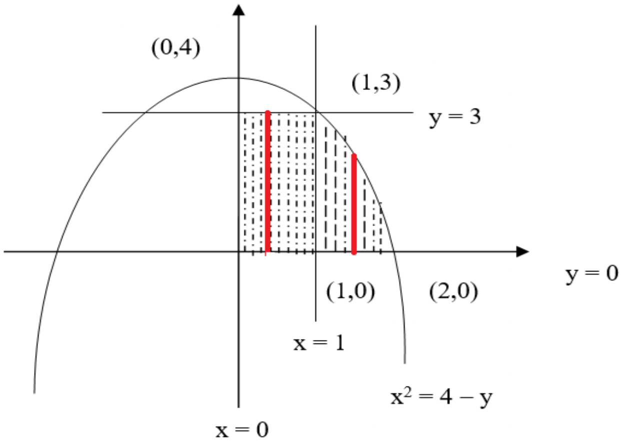
$$= \left[\frac{3x^2}{2} + \frac{9x}{2} \right]_0^1 + \frac{241}{60}$$

$$= \frac{3}{2} + \frac{9}{2} + \frac{241}{60}$$

$\therefore$ the region of integration is bounded by

$y = 0 \qquad x = 0$

$y = 3 \qquad x = \sqrt{4-y} \quad i.e. \quad x^2 = 4 - y$



In the first region x varies from $0 \; to \; 1$. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on $y = 0$ and upper end lies on $y = 3$.

In the second region x varies from $1 \; to \; 2$. To find the limit for y, we take a strip parallel to the y – axis, it's lower end lies on $y = 0$ and upper end lies on $x^2 = 4 - y; \quad y = 4 - x^2$

17. Evaluate $\int_{1}^{3} \int_{y=0}^{\frac{6}{x}} x^2 \, dy\, dx$ by change the order of integration.

Solution: Given integral is in proper form.

By changing the order, we have

$$I = \int_0^2 \int_1^3 x^2 \; dx \; dy + \int_2^6 \int_1^{\frac{6}{y}} x^2 \; dx \; dy$$

$$= \int_0^2 \left[\frac{x^3}{3}\right]_1^3 dy + \int_2^6 \left[\frac{x^3}{3}\right]_1^{\frac{6}{y}} dy$$

$$= \int_0^2 \left[9-\frac{1}{3}\right] dy + \frac{1}{3}\int_2^6 \left[\frac{216}{y^3}-\frac{1}{3}\right] dy$$

$$= \frac{26}{3}\left[y\right]_0^2 + \frac{1}{3}\left[-\frac{216}{2y^2}-\frac{y}{3}\right]_2^6$$

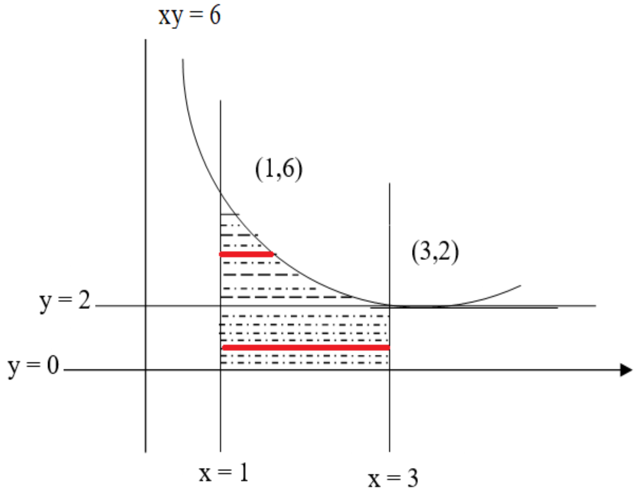
$$= \frac{52}{3} + \frac{1}{3}\left[\left(-3-2\right)-\left(-27-\frac{2}{3}\right)\right]$$

$$= \frac{52}{3} + \frac{68}{9}$$

$\therefore$ the region of integration is bounded by

$x=1 \qquad y=0$

$x=3 \qquad y=\frac{6}{x} \quad i.e. \quad xy=6$



In the bottom region y varies from $0 \; to \; 2$. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $x=1$ and right end lies on $x=3$.

In the top region y varies from $2 \; to \; 6$. To find the limit for x, we take a strip parallel to the x – axis, it's left end lies on $x=1$ and right end lies on $xy=6; \quad x=\frac{6}{y}$.

Exercise

1. Change the order of integration and then evaluate the following integrals:

(i) $\int_0^1 \int_x^{\sqrt{2-x^2}} dy\,dx$ (ii) $\int_1^2 \int_0^{\frac{4}{x}} xy \; dy\,dx$ (iii) $\int_0^1 \int_{y^2}^{y} \frac{y}{x^2+y^2}\,dy\,dx$ (iv) $\int_0^3 \int_1^{\sqrt{4-x}} x+y \; dy\,dx$

Change of variable in double integral

Quite often, the evaluation of a double integral is greatly simplified by a suitable change of variables. Let the variables x, y in the double integral $\iint_A f(x,y)dxdy$ be changed to u, v by means of the relations $x=\phi(u,v),\ y=\psi(u,v)$ then the double integral is transformed to $\iint_A f(\phi(u,v),\psi(u,v))\,|J|\,du\,dv$ where $J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$ and A is the region in the $uv-$ plane which corresponds to the area A in the $xy-$plane.

Change to polar co-ordinates

Let $x=r\cos\theta,\ y=r\sin\theta$ *then* $x^2+y^2=r^2$ *and* $dxdy=|J|\,drd\theta=rdrd\theta$.

Because $J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}x_r & x_\theta\\ y_r & y_\theta\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\left(\cos^2\theta+\sin^2\theta\right)=r$

The limits for r *and* θ can be found by the region of integration. It should be noted that the change from Cartesian to polar co-ordinate is useful when the region of integration is a circle or part of a circle.

1. Evaluate $\int_0^a\int_y^a \frac{x}{x^2+y^2}dy\,dx$ changing into polar coordinates

Rewriting the given integral in proper order, we have

$$\int_0^a\int_y^a \frac{x}{x^2+y^2}dx\,dy$$

Let

$x=r\cos\theta,\ y=r\sin\theta$ *then* $x^2+y^2=r^2$ *and* $dxdy=rdrd\theta$

$$I=\int_0^{\frac{\pi}{4}}\int_0^{a\sec\theta}\frac{r\cos\theta}{r^2}\,rdr\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\int_0^{a\sec\theta}\cos\theta\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}(\cos\theta)(a\sec\theta)\,d\theta$$

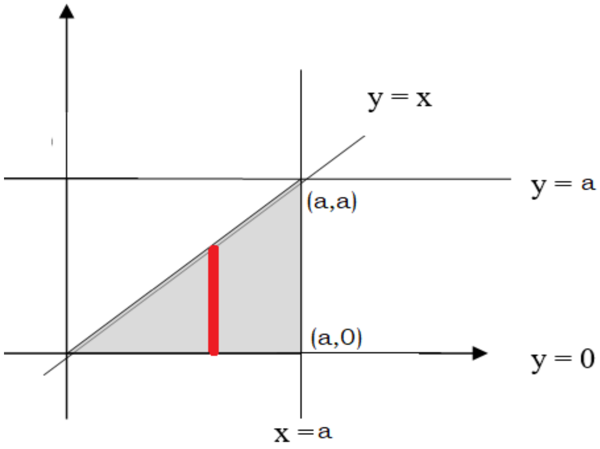
$$=\int_0^{\frac{\pi}{4}}a\,d\theta$$

$$=\frac{\pi}{4}\times a$$

$\therefore$ the region of integration is bounded by

$x=y \qquad y=0$

$x=a \qquad y=a$



Here r varies from 0 to the line x = a. i.e. $r\cos\theta=a$ *i.e.* $r=a\sec\theta$ and θ varies from 0 to π/4.

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2 Evaluate $\int\limits_0^2 \int\limits_0^{\sqrt{4-y^2}} x^2+y^2\ dx\,dy$ **by changing into polar coordinates.**

Let

$x = r\cos\theta,\ \ y = r\sin\theta \ then\ x^2+y^2 = r^2 \ and \ dxdy = rdrd\theta$

$$\int\limits_0^2 \int\limits_0^{\sqrt{4-y^2}} x^2+y^2\ dx\,dy = \int\limits_0^{\frac{\pi}{2}} \int\limits_0^2 r^2\ \ r\,dr\,d\theta$$

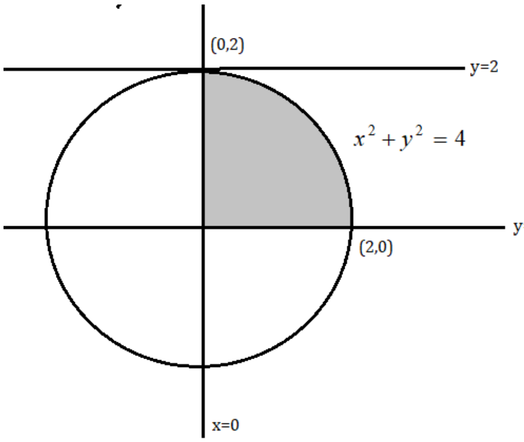
$$= \int\limits_0^{\frac{\pi}{2}} \int\limits_0^2 r^3\ \ dr\,d\theta$$

$$= \int\limits_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^2 d\theta$$

$$= \int\limits_0^{\frac{\pi}{2}} 4d\theta \ \ = \ 4\frac{\pi}{2} = 2\pi$$

Given limits are y = 0 & y = 2
x = 0 &

$x = \sqrt{4-y^2}$ i.e. $x^2 = 4-y^2$ i.e. $x^2+y^2 = 4$



Here r varies from 0 to 2 and θ varies from 0 to π/2.

.

3 Using polar coordinates, evaluate $\iint\limits_R e^{x^2+y^2} dydx$ **, where** R **is the semi circular region bounded by the** $x-$ **axis and the curve** $y = \sqrt{1-x^2}$ **.**

Let

$x = r\cos\theta,\ \ y = r\sin\theta \ then\ x^2+y^2 = r^2 \ and \ dxdy = rdrd\theta$

$$\iint\limits_R e^{x^2+y^2} dydx = \int\limits_0^{\pi} \int\limits_0^1 e^{r^2}\ r\ dr\ d\theta$$

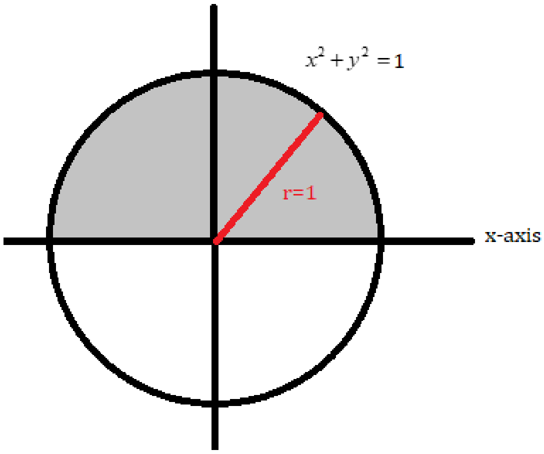
$$= \frac{1}{2}\int\limits_0^{\pi} \int\limits_0^1 e^{r^2}\ \ d\left(r^2\right)\ d\theta$$

$$= \frac{1}{2}\int\limits_0^{\pi} \left[e^{r^2}\right]_0^1 \ \ d\theta$$

$$= \frac{1}{2}\left[e^1 - 1\right] \int\limits_0^{\pi} \ \ d\theta$$

$$= \frac{\pi}{2}\left[e-1\right]$$

The area is bounded by x-axis and the curve $y = \sqrt{1-x^2}$ i.e. $x^2+y^2 = 1$.



Here r varies from 0 to 1 and θ varies from 0 to π.

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4 Evaluate $\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy$ **by changing into polar coordinates.**

Let

$$x=r\cos\theta,\ \ y=r\sin\theta\ then\ x^2+y^2=r^2$$
$$and\ \ dxdy=rdrd\theta$$

$$\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-\left(r^2\right)}\ r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-u}\ \ du\ \ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta\ = \frac{1}{2}\,\frac{\pi}{2}$$

Given limits are y = 0 & y = ∞

x = 0 & x = ∞

Here r varies from 0 to ∞ & θ varies from 0 to π/2.

$$Let\ \ r^2=u,\ \ \ 2r\,dr=du$$

$$when\ r=0,\ u=0\ \ and\ \ when\ \ r=\infty,\ u=\infty$$

$$Also\ \int_0^\infty e^{-u}\,du=1$$

5 Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ **by changing into polar coordinates.**

Given limits are x = 0 & x = 2

$$y=\sqrt{2x-x^2}$$

y = 0 & $y^2=2x-x^2$

$$x^2+y^2-2x=0$$

$$x=r\cos\theta,\ \ y=r\sin\theta\ then\ x^2+y^2=r^2\ ,\ dxdy=rdrd\theta$$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx = \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta}{r}\ r\,dr\,d\theta$$

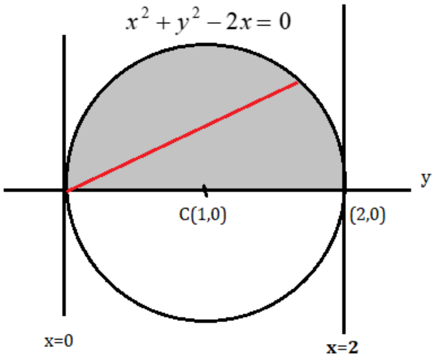
$$= \int_0^{\pi/2} \int_0^{2\cos\theta} r\cos\theta\ \ dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^{2\cos\theta} \cos\theta\ \ \ d\theta$$

$$= 2\int_0^{\pi/2} \cos^3\theta\ \ d\theta$$

$$= \ 2\,\frac{3-1}{3}.1\ \ = \frac{4}{3}$$

This is a circle with centre (1,0) and radius 1.



$$r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta = 0$$

$$r^2 - 2r\cos\theta = 0$$

$$r\left(r-2\cos\theta\right)=0$$

$$r=0,\ \ r=2\cos\theta$$

Also θ varies from 0 to π/2.

6 Evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} \left(x^2+y^2\right) dy\,dx$ **by changing into polar coordinates.**

Let

$x = r\cos\theta,\ \ y = r\sin\theta \ then\ x^2+y^2 = r^2$

$and\ \ dxdy = rdrd\theta$

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} x^2+y^2\ dy\,dx = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{16a^4}{4}\int_0^{\pi/2}\cos^4\theta\ d\theta$$

$$= 4a^4\ \frac{4-1}{4}.\frac{4-3}{4-2}\ \frac{\pi}{2}$$

$$= \frac{3}{4}\pi a^4$$

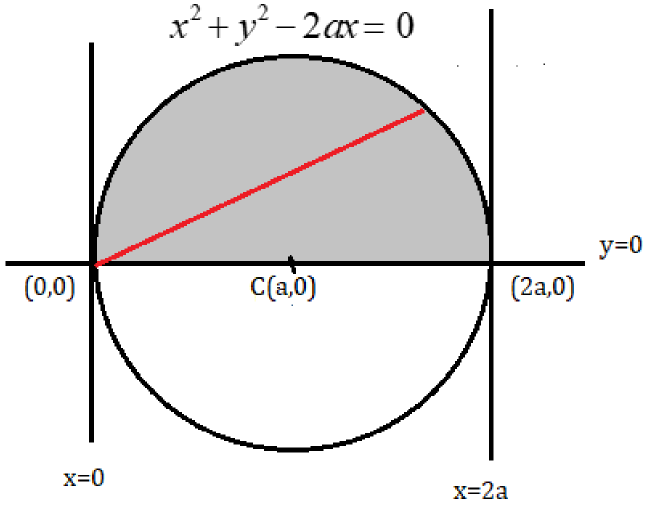
Given limits are $x = 0$ & $x = 2a$

$y = 0$ & $y = \sqrt{2ax-x^2}$

$i.e.\ \ y^2 = 2ax - x^2$

$i.e.\ \ \ x^2+y^2-2ax = 0$

This is a circle with centre (a,0) and radius a.



$$r^2\cos^2\theta + r^2\sin^2\theta - 2ar\cos\theta = 0$$

$$r^2 - 2ar\cos\theta = 0$$

$$r\left(r - 2a\cos\theta\right) = 0$$

$$r = 0,\ \ r = 2a\cos\theta$$

Also θ varies from 0 to π/2.

7 Evaluate $\iint xy\left(x^2+y^2\right)^{\frac{3}{2}}dxdy$ over the positive quadrant of the circle $x^2+y^2=1$.

$$\iint xy\left(x^2+y^2\right)^{\frac{3}{2}}dxdy=\int_0^{\frac{\pi}{2}}\int_0^1 r\cos\theta.r\sin\theta.\left(r^2\right)^{\frac{3}{2}}\ r\ dr\ d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_0^1 r^6\cos\theta.\sin\theta\ dr\ d\theta$$

$$=\frac{1}{7}\int_0^{\frac{\pi}{2}}\left[r^7\right]_0^1\cos\theta.\sin\theta\ d\theta$$

$$=\frac{1}{7}.\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin 2\theta\ d\theta$$

$$=\frac{1}{14}\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

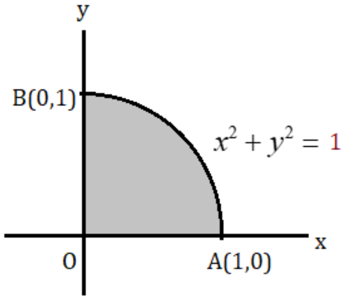
$$=-\frac{1}{2}.\frac{1}{14}[-1-1]\ =\ \frac{1}{14}$$

Let

$x=r\cos\theta,\ y=r\sin\theta\ then$

$x^2+y^2=r^2\ and$

$dxdy=rdrd\theta$



The region of integration is given here in which r varies from $0\ to\ 1$ while θ varies from $0\ to\ \frac{\pi}{2}$.

Exercise

1 Evaluate the following integrals by changing to polar coordinates:

(i) $\iint\frac{\sqrt{1-x^2-y^2}}{\sqrt{1+x^2+y^2}}dxdy$ over the positive quadrant of the unit circle

(ii) $\iint\sqrt{a^2-x^2-y^2}dxdy$ over the semicircle $x^2+y^2=ax$ in the positive quadrant

(iii) $\iint\left(a^2-x^2-y^2\right)dxdy$ over the semicircle $x^2+y^2=ax$ in the positive quadrant

(iv) $\iint\frac{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}{\sqrt{1+\frac{x^2}{a^2}+\frac{y^2}{b^2}}}dxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(Hint: By suitable substitution convert this into circular region and then apply polar coordinates)

2 Evaluate $\int_0^a\int_y^a\frac{x^2}{\sqrt{x^2+y^2}}dxdy$ by changing to polar coordinates.

3 Transform the integral $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}\frac{1}{\sqrt{a^2-x^2-y^2}}dy\,dx$ in polar coordinates and then evaluate it.

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Evaluation of Triple Integrals

Consider the triple integral $\int\limits_{z_1}^{z_2}\int\limits_{y_1}^{y_2}\int\limits_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$.

a. If $x_1, x_2, y_1, y_2, z_1, z_2$ are constants, then the order of integration is immaterial, provided the limits of integration are changed accordingly.

b. If z_1, z_2 are functions of x & y; y_1, y_2 are functions of x; x_1, x_2 are constants, then the integration is to be performed firstly w.r.t z, then w.r.t y and finally w.r.t. x. Thus $\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}\int\limits_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx$.

c. If $f(x,y,z)=1$, then the triple integral $\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}\int\limits_{z_1}^{z_2} dz\,dy\,dx$ gives the volume enclosed by the regions

1 Evaluate $\int\limits_0^a\int\limits_0^b\int\limits_0^c x^2+y^2+z^2\,dz\,dy\,dx$

Let $I=\int\limits_0^a\int\limits_0^b\int\limits_0^c x^2+y^2+z^2\,dz\,dy\,dx$

$$=\int\limits_0^a\int\limits_0^b \left[x^2z+y^2z+\frac{z^3}{3}\right]_0^c dy\,dx$$

$$=\int\limits_0^a\int\limits_0^b \left[x^2c+y^2c+\frac{c^3}{3}\right] dy\,dx$$

$$=\int\limits_0^a \left[x^2cy+\frac{y^3}{3}c+\frac{c^3}{3}y\right]_0^b dx$$

$$=\int\limits_0^a \left[x^2cb+\frac{b^3}{3}c+\frac{c^3}{3}b\right] dx$$

$$=\left[\frac{x^3}{3}cb+\frac{b^3}{3}cx+\frac{c^3}{3}bx\right]_0^a$$

$$=\left[\frac{a^3}{3}cb+\frac{b^3}{3}ca+\frac{c^3}{3}ba\right] \quad =\frac{abc}{3}\left[a^2+b^2+c^2\right]$$

2 Evaluate $\int\limits_0^a\int\limits_0^b\int\limits_0^c e^{x+y+z}\,dz\,dy\,dx$

Let $I=\int\limits_0^a\int\limits_0^b\int\limits_0^c e^{x+y+z}\,dz\,dy\,dx$

$$=\int\limits_0^a e^x dx\int\limits_0^b e^y dy\int\limits_0^c e^z dz$$

$$=\left(e^x\right)_0^a\left(e^y\right)_0^b\left(e^z\right)_0^c$$

$$=\left(e^a-1\right)\left(e^b-1\right)\left(e^c-1\right)$$

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3 Evaluate: $\int\limits_1^e \int\limits_0^{\log y} \int\limits_1^{e^x} \log z \; dz\; dx\; dy$

Let $I = \int\limits_1^e \int\limits_0^{\log y} \int\limits_1^{e^x} \log z \; dz\; dx\; dy$

$$= \int\limits_1^e \int\limits_0^{\log y} (x-1)e^x + 1 \; dx\; dy \quad \text{\{Refer next column\}}$$

$$\boxed{\int uv = (u)(v_1) - (u')(v_2) + \ldots}$$

$$= \int\limits_1^e \left[(x-1)e^x - (1)e^x + x\right]_0^{\log y} dy$$

$$= \int\limits_1^e \left[xe^x - 2e^x + x\right]_0^{\log y} dy$$

$$= \int\limits_1^e y\log y - 2y + \log y + 2 \; dy$$

$$= \int\limits_1^e (y+1)\log y + (2-2y)\; dy$$

$$= \int\limits_1^e (y+1)\log y \; dy + \int\limits_1^e (2-2y)\; dy$$

$$= \frac{e^2}{4} + \frac{5}{4} + \left(2y - y^2\right)_1^e \quad \text{\{Refer next column\}}$$

$$= \frac{e^2}{4} + \frac{5}{4} + \left(2e - e^2 - 1\right)$$

$$= 2e - \frac{3}{4}e^2 + \frac{1}{4}$$

Consider $I = \int\limits_1^{e^x} \log z \,.\, 1 \; dz$

Let

$u = \log z \quad and \quad dv = 1.dz$

$du = \frac{1}{z}dz \;\; and \quad v = z$

$$I = \left[z.\log z\right]_1^{e^x} - \int\limits_1^{e^x} z \, \frac{1}{z} \, dz$$

$$= e^x x - \left(e^x - 1\right)$$

$$= (x-1)e^x + 1$$

Consider $I = \int\limits_1^e (y+1)\log y \; dy$

$u = \log y \quad and \quad dv = (y+1)dy$

$du = \frac{1}{y}dy \;\; and \quad v = \frac{y^2}{2} + y$

$$I = \left[\left(\frac{y^2}{2} + y\right)(\log y)\right]_1^e - \int\limits_1^e \left(\frac{y^2}{2} + y\right)\frac{1}{y}dy$$

$$= \left[\left(\frac{e^2}{2} + e\right)(\log e)\right] - \int\limits_1^e \left(\frac{y}{2} + 1\right)dy$$

$$= \left[\left(\frac{e^2}{2} + e\right)\right] - \left[\frac{y^2}{4} + y\right]_1^e$$

$$= \left[\left(\frac{e^2}{2} + e\right)\right] - \left[\frac{e^2}{4} + e - \frac{1}{4} - 1\right]$$

$$= \frac{e^2}{4} + \frac{5}{4}$$

.

4 Evaluate $\int_0^a \int_0^b \int_0^c xyz\,dz\,dy\,dx$

$$Given\ \ I = \int_0^a x\,dx \int_0^b y\,dy \int_0^c z\,dz$$

$$= \left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b \left[\frac{z^2}{2}\right]_0^c$$

$$= \left[\frac{a^2}{2}\right]\left[\frac{b^2}{2}\right]\left[\frac{c^2}{2}\right] \quad = \frac{(abc)^2}{8}$$

5 Evaluate $\int_0^1 \int_1^2 \int_2^3 xy^2z\,dz\,dy\,dx$

$$\int_0^1 \int_1^2 \int_2^3 xy^2z\,dz\,dy\,dx = \int_0^1 x\,dx \int_1^2 y^2\,dy \int_2^3 z\,dz$$

$$= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^3}{3}\right]_1^2 \left[\frac{z^2}{2}\right]_2^3$$

$$= \left(\frac{1}{2}\right)\left(\frac{8}{3}-\frac{1}{3}\right)\left(\frac{9}{2}-\frac{4}{2}\right) \quad = \frac{35}{12}$$

.

6 Evaluate $\int_0^1 \int_0^2 \int_0^3 xy^2z\,dz\,dy\,dx$

$$\int_0^1 \int_0^2 \int_0^3 xy^2z\,dz\,dy\,dx = \int_0^1 x\,dx \int_0^2 y^2\,dy \int_0^3 z\,dz$$

$$= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^3}{3}\right]_0^2 \left[\frac{z^2}{2}\right]_0^3$$

$$= \left(\frac{1}{2}-0\right)\left(\frac{8}{3}-0\right)\left(\frac{9}{2}-0\right)$$

$$= 6$$

7 Evaluate $\int_0^{2a} \int_0^x \int_y^x xyz\,dz\,dy\,dx$

$$\int_0^{2a} \int_0^x \int_y^x xyz\,dz\,dy\,dx = \int_0^{2a} \int_0^x xy\left[\frac{z^2}{2}\right]_y^x dy\,dx$$

$$= \frac{1}{2} \int_0^{2a} \int_0^x xy\left(x^2 - y^2\right) dy\,dx$$

$$= \frac{1}{2} \int_0^{2a} \int_0^x \left(x^3y - xy^3\right) dy\,dx$$

$$= \frac{1}{2} \int_0^{2a} \left(x^3\frac{y^2}{2} - x\frac{y^4}{4}\right)_0^x dx$$

$$= \frac{1}{2} \int_0^{2a} \left(\frac{x^5}{2} - \frac{x^5}{4}\right) dx$$

$$= \frac{1}{2} \left(\frac{x^6}{12} - \frac{x^6}{24}\right)_0^{2a}$$

$$= \frac{1}{2} \left(\frac{64a^6}{12} - \frac{a^6}{24}\right)$$

$$= \frac{1}{2}a^6 \left(\frac{64}{12} - \frac{1}{24}\right)$$

$$= \frac{1}{2}a^6 \left(\frac{127}{24}\right)$$

$$= \frac{127}{48}a^6$$

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8 Evaluate : $\iiint\limits_{x^2+y^2+z^2\le 1} z^2 dxdydz$

The region R is bounded by the sphere

$x^2+y^2+z^2\le 1$ can be expressed as

$-1\le x\le 1, \quad -\sqrt{1-x^2}\le y\le \sqrt{1-x^2},$

$-\sqrt{1-x^2+y^2}\le z\le \sqrt{1-x^2+y^2}$

$$\iiint\limits_{R} z^2 dxdydz=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} z^2\ dz\ dy\ dx$$

$$=2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z^2\ dz\ dy\ dx$$

{Because z^2 is even function}

$$=\frac{2}{3}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left[z^3\right]_0^{\sqrt{1-x^2-y^2}}\ dz\ dy\ dx$$

$$=\frac{2}{3}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(1-x^2-y^2\right)^{\frac{3}{2}}\ dy\ dx$$

$$=\frac{4}{3}\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\left(1-x^2-y^2\right)^{\frac{3}{2}}\ dy\ dx$$

{since the integrand y^2 is even}

$Put\ \ y=\sqrt{1-x^2}\sin\theta\ \ \ hence\ \ \ dy=\sqrt{1-x^2}\cos\theta\ d\theta$

$When\ \ \ y=0, \quad and \quad when\ \ y=\sqrt{1-x^2}$

$0=\sqrt{1-x^2}\sin\theta \qquad \sqrt{1-x^2}=\sqrt{1-x^2}\sin\theta$

$0=\sin\theta \qquad 1=\sin\theta$

$\therefore\ \theta=0 \qquad \theta=\dfrac{\pi}{2}$

$$=\frac{4}{3}\int_{-1}^{1}\int_{0}^{\frac{\pi}{2}}\left[\left(1-x^2\right)\cos^2\theta\right]^{\frac{3}{2}}\sqrt{1-x^2}\cos\theta\ d\theta\ dx$$

$$=\frac{4}{3}\int_{-1}^{1}\int_{0}^{\frac{\pi}{2}}\left(1-x^2\right)^2\cos^4\theta\ d\theta\ dx$$

$$=\frac{4}{3}\int_{-1}^{1}\left(1-x^2\right)^2\frac{3\times 1}{4\times 2}\frac{\pi}{2}\ dx$$

$$=\frac{4}{3}\frac{3\pi}{16}\int_{-1}^{1} 1+x^4-2x^2\ dx$$

$$=\frac{\pi}{2}\int_{0}^{1} 1+x^4-2x^2\ dx$$

{since the integrand is even}

$$=\frac{\pi}{2}\left[x+\frac{x^5}{5}-\frac{2x^3}{3}\right]_0^1$$

$$=\frac{\pi}{2}\left[1+\frac{1}{5}-\frac{2}{3}\right]=\frac{4\pi}{15}$$

.

9 Evaluate

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz\,dy\,dx$$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz\,dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\ dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}1 - \sin^{-1}0\right] dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2}\, dy\,dx$$

$$= \frac{\pi}{2}\int_0^a \sqrt{a^2-x^2}\ dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{\pi}{2}\frac{a^2}{2}\sin^{-1}1$$

$$= \frac{\pi}{2}\frac{a^2}{2}\frac{\pi}{2} = \frac{\pi^2 a^2}{8}$$

10 Evaluate $\iiint (x+y+z)\ dx\ dy\ dz$ over the tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$.

From the given data, the limit of the region is expressed as $0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y$.

$$I = \iiint (x+y+z)\ dx\ dy\ dz$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\ dz\ dy\ dx$$

$$= \int_0^1 \int_0^{1-x} \left[(x+y)z + \frac{z^2}{2}\right]_0^{1-x-y} dy\ dx$$

$$= \int_0^1 \int_0^{1-x} \left[(x+y)(1-x-y) + \frac{(1-x-y)^2}{2}\right] dy\ dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (1-x-y)(1+x+y)\ dy\ dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} \left[1-(x+y)^2\right] dy\ dx$$

$$= \frac{1}{2}\int_0^1 \left[y - \frac{(x+y)^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[(1-x) - \frac{1}{3} + \frac{x^3}{3}\right] dx$$

$$= \frac{1}{2}\left[x - \frac{x^2}{2} - \frac{x}{3} + \frac{x^4}{12}\right]_0^1$$

$$= \frac{1}{2}\left[1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{12}\right] = \frac{1}{8}$$

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11 Evaluate $\iiint \frac{1}{(x+y+z+1)^3}\, dx\ dy\ dz$ **over the region** $x\ge 0,\ y\ge 0,\ z\ge 0,\ x+y+z\le 1.$

(Note: over the region bounded by the coordinate planes and the plane $x+y+z=1$.)

From the given data, the limit of the region is expressed as
$0\le x\le 1,\ 0\le y\le 1-x,\ 0\le z\le 1-x-y.$

$$\text{I} = \iiint \frac{1}{(x+y+z+1)^3}\, dx\ dy\ dz$$

$$=\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^{-3}\ dz\ dy\ dx$$

$$=\int_0^1 \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\ dx$$

$$=\int_0^1 \int_0^{1-x} \left[-\frac{1}{8}+\frac{1}{2(x+y+1)^2}\right] dy\ dx$$

$$=\int_0^1 \left[-\frac{y}{8}-\frac{1}{2(x+y+1)}\right]_0^{1-x} dx$$

$$=\int_0^1 \ -\frac{1-x}{8}-\frac{1}{4}+\frac{1}{2(x+1)}\ dx$$

$$=\left[-\frac{1}{8}\frac{(1-x)^2}{-2}-\frac{x}{4}+\frac{1}{2}\log(x+1)\right]_0^1$$

$$=-\frac{1}{4}+\frac{1}{2}\log 2-\frac{1}{16}$$

$$=\frac{1}{2}\log 2-\frac{5}{16}$$

12 Find the volume of the sphere $x^2+y^2+z^2=a^2$ **using triple integration.**

Volume of the Sphere = 8 (Volume in the first octant)

$$= 8 \iiint_V dxdydz$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\ dy\ dx$$

$$= 8 \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{\left(a^2-x^2\right)}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8 \int_0^a \left[\frac{\left(a^2-x^2\right)}{2}\sin^{-1}1\right] dx$$

$$= \frac{8}{2}\frac{\pi}{2}\int_0^a \left(a^2-x^2\right) dx$$

$$= \frac{8}{2}\frac{\pi}{2}\left[a^2x-\frac{x^3}{3}\right]_0^a = \frac{4}{3}\pi a^3$$

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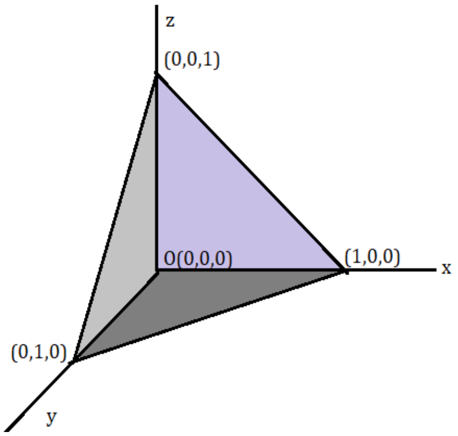
13 Find the volume of the tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$.



From the given data, the limit of the region is expressed as $0\le x\le 1,\ 0\le y\le 1-x,\ 0\le z\le 1-x-y.$ Therefore volume of tetrahedron is $V=\iiint\limits_R dx\,dy\,dz$

$$=\int_0^1\int_0^{1-x}\int^{1-x-y} dz\,dy\,dx$$

$$=\int_0^1\int_0^{1-x} 1-x-y\,dy\,dx$$

$$=\int_0^1\left[y-xy-\frac{y^2}{2}\right]_0^{1-x}dx$$

$$=\int_0^1\left[(1-x)-x(1-x)-\frac{(1-x)^2}{2}\right]dx$$

$$=\left[\frac{(1-x)^2}{-2}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{(1-x)^3}{-6}\right]_0^1$$

$$=\left[-\frac{1}{2}+\frac{1}{3}+\frac{1}{2}-\frac{1}{6}\right]=\frac{1}{6}$$

Exercise

1 Evaluate the following triple integrals:

(i) $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z} x+y+z\ dydxdz$ (ii) $\int_0^a\int_0^a\int_0^a xy+yz+xz\ dxdydz$ (iii) $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta$

2 Evaluate $\iiint xyz\,dxdydz$ over the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{z^2}=1$

3. Evaluate $\iiint\limits_V xy+yz+xz\ dxdydz$ where V is bounded by $x=0,\ x=1,\ y=0,\ y=2,\ z=0\ and\ z=3.$

4. Evaluate $\iiint\limits_V \frac{1}{(x+y+z+1)^3}\ dxdydz$ where V is bounded by $x=0,\ y=0,\ z=0\ and\ x+y+z=1.$

5. Evaluate $\iiint\limits_V xyz\ dxdydz$ where V is the region of space bounded by the coordinate planes $x=0,\ y=0,\ z=0$ and the sphere $x^2+y^2+z^2=1$ and contained in the positive octant.

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Change of variables in a triple integral (Change to cylindrical polar co-ordinates)

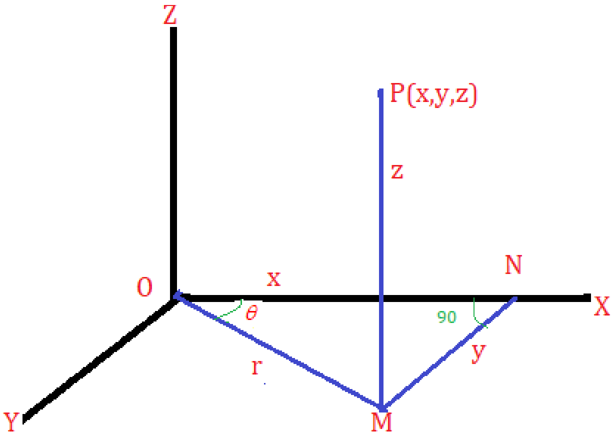
Let $x = r\cos\theta,\ \ y = r\sin\theta\ and\ \ z = z$

$Then\ x^2 + y^2 = r^2\ and\ dxdydz = |J|\,drd\theta dz = r\ drd\theta dz$

$$\text{Because } J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\times\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

The limits for $r\ and\ \theta$ and z can be found from the region of integration and $\theta \in (0, 2\pi)$.

Cylindrical polar coordinates are useful when the region of integration is a right circular cylinder



1 Evaluate $\displaystyle\iiint\limits_{\substack{x^2+y^2\le 1\\ 2\le z\le 3}} z\left(x^2+y^2\right)dx\,dy\,dz$

Changing to cylindrical polar co-ordinates by the relations $x = r\cos\theta,\ y = r\sin\theta\ and\ \ z = z$ the region $\left\{(x,y,z): x^2 + y^2 \le 1;\ 2 \le z \le 3\right\}$ is transformed to the region $\{(r,\theta,z): 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 2 \le z \le 3\}$. Here $dxdydz$ is to be replaced by $r\ dr\ d\theta\ dz$ and $x^2 + y^2$ is replaced by r^2.

$$\iiint\limits_{\substack{x^2+y^2\le 1\\ 2\le z\le 3}} z\left(x^2+y^2\right)\,dx\,dy\,dz = \int_2^3\int_0^{2\pi}\int_0^1 z.r^2.r\,dr\,d\theta\,dz$$

$$=\int_2^3\int_0^{2\pi} z\left[\frac{r^4}{4}\right]_0^1 d\theta\,dz$$

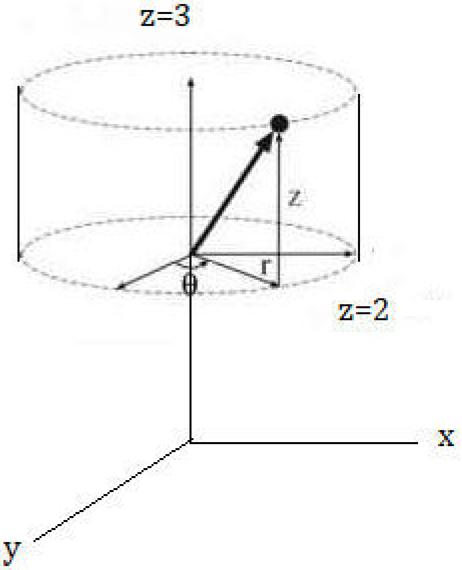
$$=\frac{1}{4}\int_2^3\int_0^{2\pi} z\,d\theta\,dz$$

$$=\frac{1}{4}\int_2^3 z\left(2\pi-0\right)dz$$

$$=\frac{2\pi}{4}\int_2^3 z\,dz$$

$$=\frac{2\pi}{4}\left[\frac{z^2}{2}\right]_2^3$$

$$=\frac{2\pi}{4}\left[\frac{9}{2}-\frac{4}{2}\right]=\frac{5\pi}{4}$$



2 Find the volume of the cylinder with base radius a and height h.

The equation of cylinder is $\left\{x^2+y^2\le a^2;\,0\le z\le h\right\}$. Required volume = $\iiint\limits_V dxdydz$

Changing to cylindrical polar co-ordinates by the relations $x=r\cos\theta,\ y=r\sin\theta\ and\ \ z=z$

$Then\ x^2+y^2=r^2\ and\ dxdydz=rdrd\theta dz$. Here $\left\{0\le r\le a,\,0\le\theta\le 2\pi,\,0\le z\le h\right\}$

Required volume $=\int_0^h\int_0^{2\pi}\int_0^a r\,dr\,d\theta\,dz$

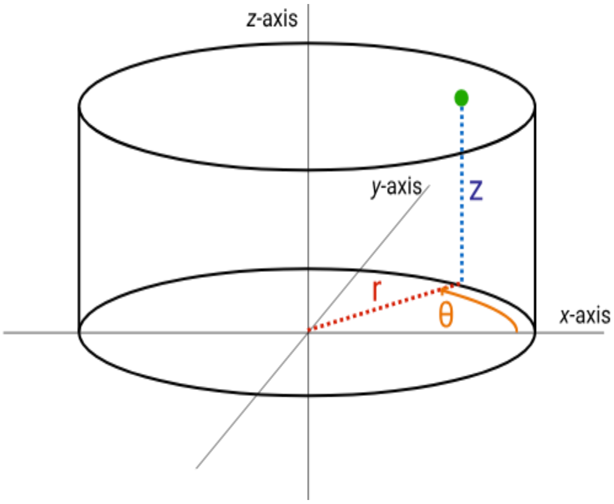
$$=\int_0^h\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^a d\theta\,dz$$

$$=\frac{a^2}{2}\int_0^h\int_0^{2\pi} d\theta\,dz$$

$$=\frac{a^2}{2}\int_0^h (2\pi-0)\,dz$$

$$=\frac{2\pi a^2}{2}\int_0^h dz$$

$$=\frac{2\pi a^2}{2}(h-0)=\pi a^2 h$$



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3 Evaluate $\iiint z\left(x^2+y^2+z^2\right)dx\,dy\,dz$ **through the volume of the cylinder** $x^2+y^2=1$ **intercepted by the planes** $z=2$ & $z=3$**.**

Changing to cylindrical polar co-ordinates by the relations $x=r\cos\theta,\ y=r\sin\theta\ and\ z=z$ the region $\left\{(x,y,z): x^2+y^2\le 1;\ 2\le z\le 3\right\}$ is transformed to the region $\{(r,\theta,z): 0\le r\le 1,\ 0\le\theta\le 2\pi,\ 2\le z\le 3\}$. Here $dxdydz$ is to be replaced by $r\ dr\ d\theta\ dz$ and x^2+y^2 is replaced by r^2.

$$\iiint\limits_{\substack{x^2+y^2\le 1\\ 2\le z\le 3}} z\left(x^2+y^2+z^2\right)dx\,dy\,dz=\int_2^3\int_0^{2\pi}\int_0^1 z\left(r^2+z^2\right)r\,dr\,d\theta\,dz$$

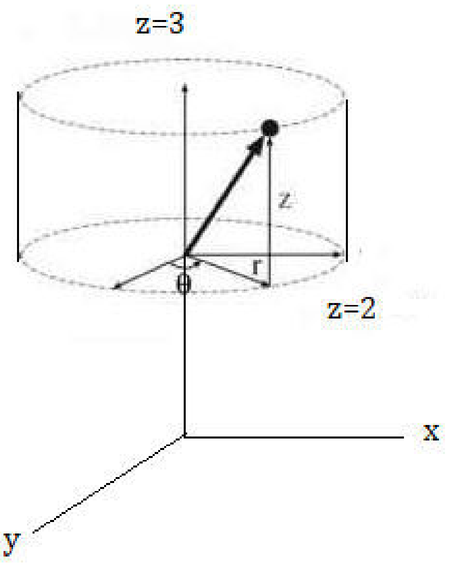
$$=\int_2^3\int_0^{2\pi}\left[z\frac{r^4}{4}+z^3\frac{r^2}{2}\right]_0^1 d\theta\,dz$$

$$=\int_2^3\int_0^{2\pi}\left[z\frac{1}{4}+z^3\frac{1}{2}\right]d\theta\,dz$$

$$=\int_2^3\left[z\frac{1}{4}+z^3\frac{1}{2}\right](2\pi-0)\ dz$$

$$=2\pi\left[\frac{z^2}{8}+\frac{z^4}{8}\right]_2^3$$

$$=2\pi\left[\frac{9}{8}+\frac{81}{8}-\frac{4}{8}-\frac{16}{8}\right]=\frac{70}{4}\pi$$



Change to spherical polar co-ordinates

The relation between the Cartesian and spherical polar co-ordinates of a point are given by the equations $x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta$ and $J=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=r^2\sin\theta\ and\ dxdydz=\left|J\right|drd\theta d\phi$. Also $x^2+y^2+z^2=r^2$. The limits for $r\ and\ \theta$ and ϕ can be found from the region of integration where $\theta\in(0,\pi)$, the angle between $OP\ and\ +ve\ z-$axis and $\phi\in(0,2\pi)$.

This is useful when the region of integration is in a part of a sphere.

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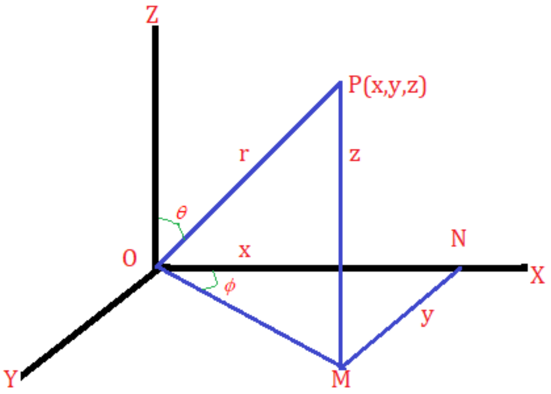


4 A point P has spherical coordinates $\left(8, \frac{2\pi}{3}, -\frac{\pi}{6}\right)$. Find the rectangular coordinates.

The rectangular coordinates are given by

$$x = r\sin\theta\cos\phi = 8\sin\left(\frac{2\pi}{3}\right)\cos\left(-\frac{\pi}{6}\right) = 8\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) = 8\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 6$$

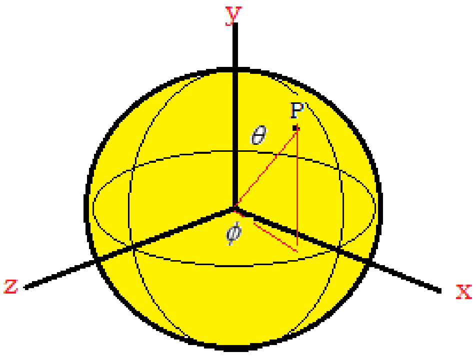
$$y = r\sin\theta\cos\phi = 8\sin\left(\frac{2\pi}{3}\right)\sin\left(-\frac{\pi}{6}\right) = -8\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) = -8\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = -2\sqrt{3}$$

$$z = r\cos\theta = 8\cos\left(\frac{2\pi}{3}\right) = 8\cos\left(\frac{\pi}{2}+\frac{\pi}{6}\right) = -8\sin\left(\frac{\pi}{6}\right) = -8\left(\frac{1}{2}\right) = -4$$

5 Find the volume of the sphere $x^2+y^2+z^2=a^2$ using spherical coordinates.

Required volume = $\iiint\limits_V dxdydz$ where V is the sphere $x^2+y^2+z^2=a^2$.

Changing to spherical polar coordinates by the relations $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$ where $0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi \quad and \quad dxdydz = r^2\sin\theta drd\theta d\phi$



$$\therefore \text{ Required Volume } = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta \; dr\; d\theta\; d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\frac{r^3}{3}\right]_0^a \sin\theta \;\; d\theta\; d\phi$$

$$= \frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi} \sin\theta \;\; d\theta\; d\phi$$

$$= \frac{a^3}{3}\int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi} \;\; d\phi$$

$$= \frac{a^3}{3}\int_0^{2\pi} \left[1+1\right] \;\; d\phi$$

$$= \frac{2a^3}{3}(2\pi - 0)$$

$$= \frac{4}{3}\pi a^3$$

6 Evaluate $\iiint \left(x^2+y^2+z^2\right) dx\; dy\; dz$ through the volume of the sphere $x^2+y^2+z^2 \le 1$.

Changing to spherical polar coordinates by the relations $x = r\sin\theta\cos\phi,\; y = r\sin\theta\sin\phi,\; z = r\cos\theta$ where $0 \le r \le a,\; 0 \le \theta \le \pi,\; 0 \le \phi \le 2\pi \;\; and \;\; dxdydz = r^2 \sin\theta drd\theta d\phi$. Also $x^2+y^2+z^2 = r^2$.

$$\iiint \left(x^2+y^2+z^2\right) dx\; dy\; dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \left(r^2\right) r^2 \sin\theta \; dr\; d\theta\; d\phi$$

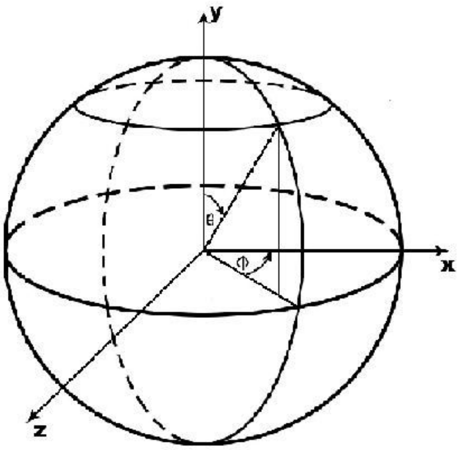
$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{r^5}{5}\right)_0^1 \sin\theta \;\; d\theta\; d\phi$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} \sin\theta \;\; d\theta\; d\phi$$

$$= \frac{1}{5}\int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi} \; d\phi$$

$$= \frac{1}{5}\int_0^{2\pi} \left[1+1\right] d\phi$$

$$= \frac{2}{5}(2\pi - 0) = \frac{4}{5}\pi$$



7 Evaluate $\iiint \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dx\, dy\, dz$ **through the volume of the positive octant of the sphere** $x^2+y^2+z^2 \le 1$.

Changing to spherical polar coordinates by the relations $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$ where $0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}$ *and* $dxdydz = r^2\sin\theta drd\theta d\phi$. Also $x^2+y^2+z^2 = r^2$.

$$\therefore \iiint \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dx\, dy\, dz$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{1} \frac{1}{\sqrt{1-r^2}}\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\cos t}\sin^2 t\,\sin\theta\,\cos t\, dt\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^2 t\,\sin\theta\ dt\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{2}.\frac{\pi}{2}\sin\theta\ d\theta\, d\phi$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \left[-\cos\theta\right]_0^{\frac{\pi}{2}}\ d\phi$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \left[0+1\right] d\phi$$

$$= \frac{\pi}{4}.\frac{\pi}{2}$$

Let

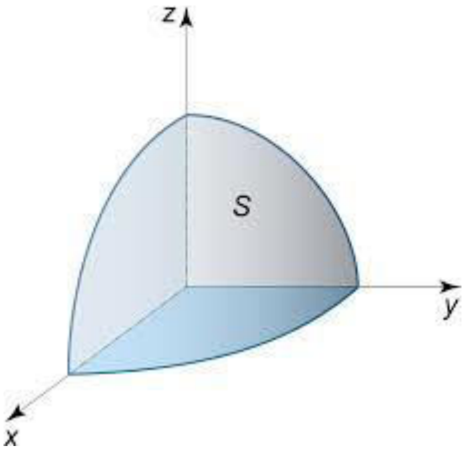
$r = \sin t$

$dr = \cos t\ dt$

when $r = 0,\ t = 0$

when $r = 1,\ t = \frac{\pi}{2}$

$\sqrt{1-r^2} = \sqrt{1-\sin^2 t} = \cos t$



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8 Evaluate $\iiint\limits_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\;dx\;dy\;dz$ **where V is**

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$

Let $I=\iiint\limits_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\;dx\;dy\;dz$

{Refer the substitutions given in (i)}. Then

$$I=\iiint\limits_{V'}\sqrt{1-u^2-v^2-w^2}\;abc\,du\,dv\,dw\;\mathbf{....}(1)$$

{changing to spherical coordinates}

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{1} abc\sqrt{1-r^2}\;r^2\sin\theta\;dr\;d\theta\;d\phi$$

{Refer the substitutions given in (2)}. Then

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{\pi}{2}} abc\;\cos t\;\sin^2 t\;\sin\theta\;\cos t\;dt\;d\theta\;d\phi$$

$$=abc\int_0^{2\pi}\int_0^{\pi}\sin\theta\int_0^{\frac{\pi}{2}}\cos^2 t\;\sin^2 t\;dt\quad d\theta\;d\phi$$

$$=abc\int_0^{2\pi}\int_0^{\pi}\sin\theta\;\frac{1.1}{4.2}.\frac{\pi}{2}\;d\theta\;d\phi$$

$$=\frac{\pi}{16}.abc\int_0^{2\pi}(-\cos\theta)_0^{\pi}\;d\phi$$

$$=\frac{\pi}{16}.abc\int_0^{2\pi}(1+1)\;d\phi$$

$$=\frac{2\pi}{16}\;abc\;(2\pi-0)$$

$$=\frac{\pi^2}{4}abc$$

Put

$\frac{x}{a}=u,\;\;\frac{y}{b}=v,\;\;\frac{z}{c}=w\;\;.......(i)$

$then\;\;dxdydz=|J|\,dudvdw$

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}x_u & x_v & x_w\\ y_u & y_v & y_w\\ z_u & z_v & z_w\end{vmatrix}$$

$$=\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=abc$$

Changing the integral (1) to spherical polar coordinates by the relations

$u=r\sin\theta\cos\phi,\;v=r\sin\theta\sin\phi,\;w=r\cos\theta$

where

$0\le r\le 1,\;0\le\theta\le\pi,\;0\le\phi\le 2\pi\;\;and$

$dudvdw=r^2\sin\theta drd\theta d\phi$

Also $u^2+v^2+w^2=r^2$.

Let

$r=\sin t\;............(ii)$

$dr=\cos t\;dt$

$when\;\;r=0,\;t=0$

$when\;\;r=1,\;t=\frac{\pi}{2}$

$\sqrt{1-r^2}=\sqrt{1-\sin^2 t}=\cos t$

Exercise

1 Evaluate $\iiint z^2 dxdydz$, by changing to cylindrical coordinates, taken over the volume bounded by the surfaces $x^2+y^2=a^2,\ x^2+y^2=z\ and\ z=0.$

2. Find the volume of the portion of the cylinder $x^2+y^2=1$ intercepted between the plane $z=0$ and the paraboloid $x^2+y^2=4-z$.

3 Find the volume of the solid surrounded by the surface $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$

(Hint: first change the variables to get a sphere and then spherical polar coordinates)

4 Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\ dz\,dy\,dx$ by transforming to spherical polar coordinates

5 Evaluate $\iiint z^2 dxdydz$, by changing to spherical coordinates, taken over the volume of the sphere $x^2+y^2+z^2\le 1$.

Evaluation of Mass

(i) Consider a plane lamina of area A with density at any arbitrary point $P(x,y)$ is $\rho=f(x,y)$.
Then its total mass M is given by $M=\iint_A \rho\ dxdy$.

In polar form , $M=\iint_A \rho\ r\ drd\theta$.

(ii) Consider a solid of volume V with density at any arbitrary point $P(x,y,z)$ is $\rho=f(x,y,z)$. Then its total mass M is given by $M=\iiint_V \rho\ dxdydz$.

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1 Find the mass of the plate whose shape is the astroid $\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}}+\frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}}=1$, given that the density of the lamina is $\rho = kxy$.

Required mass = 4 (mass in the first quadrant)

In the region $OABO$, x varies from $0 \; to \; a$ and y varies from $0 \; to \; y_1$

$$Mass = 4\int_0^a \int_0^{y_1} \rho \; dy \; dx$$

$$= 4k\int_0^a \int_0^{y_1} xy \; dy \; dx$$

$$= 4k\int_0^a x\left[\frac{y^2}{2}\right]_0^{y_1} dx$$

$$= \frac{4k}{2}\int_0^a x \; y_1^2 dx$$

$$= \frac{4k}{2}\int_0^a x \; b^2\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^3 dx$$

$$= 2kb^2\int_0^{\frac{\pi}{2}} a\sin^3\theta\left(1-\sin^2\theta\right)^3 3a\sin^2\theta\cos\theta \; d\theta$$

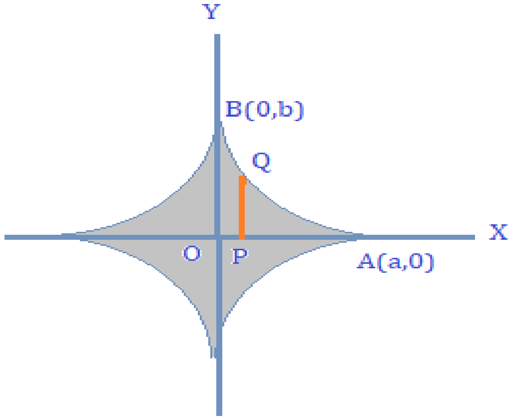
$$= 6ka^2b^2\int_0^{\frac{\pi}{2}} \sin^5\theta\left(\cos^2\theta\right)^3 \; \cos\theta \; d\theta$$

$$= 6ka^2b^2\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^7\theta \; d\theta$$

$$= 6ka^2b^2\left[\frac{4.2. \; 6.4.2}{12.10.8.6.4.2}\right]$$

$$= \frac{ka^2b^2}{20}$$

.



From the given curve

$$\frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}}=1-\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}}$$

$$y^{\frac{2}{3}}=b^{\frac{2}{3}}\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]$$

$$y=b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}=y_1$$

Put

$x = a\sin^3\theta, \; then \; dx = 3a\sin^2\theta\cos\theta \; d\theta$

$when \; x=0, \; \theta=0$

$when \; x=a, \; a=a\sin^3\theta, \; i.e. \; 1=\sin\theta, \; i.e. \; \theta=\frac{\pi}{2}$

2 Find the mass of the plate bounded by the curves $y = e^x$, $y = 0$, $x = 0$ *and* $x = 1$, whose density varies as the square of the distance from the origin.

Given that density $\rho = k\left(x^2 + y^2\right)$.

$$Mass = \int_0^1 \int_0^{e^x} \rho \; dy \; dx$$

$$= k \int_0^1 \int_0^{e^x} x^2 + y^2 \; dy \; dx$$

$$= k \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{e^x} dx$$

$$= k \int_0^1 \left[x^2 e^x + \frac{\left(e^x\right)^3}{3} \right] dx$$

$$= k \int_0^1 \left[x^2 e^x + \frac{e^{3x}}{3} \right] dx$$

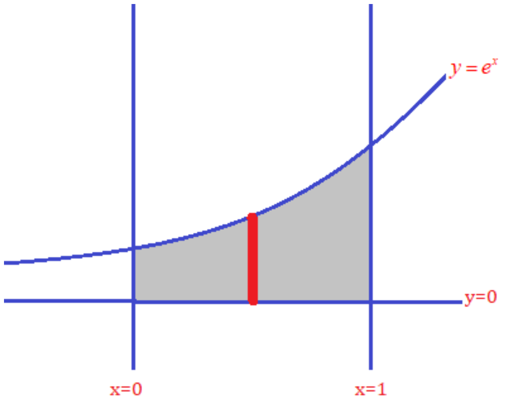
$$= k \left[\left(x^2\right)(e^x) - (2x)(e^x) + (2)(e^x) + \frac{e^{3x}}{9} \right]_0^1$$

$$= k \left[x^2 e^x - 2xe^x + 2e^x + \frac{e^{3x}}{9} \right]_0^1$$

$$= k \left[e - 2e + 2e + \frac{e}{9} - 2 - \frac{1}{9} \right]$$

$$= k \left[\frac{10e}{9} - \frac{19}{9} \right]$$

.



Let $P(x, y)$ be any point in the plate. Therefore the distance from the origin O to P is $OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$

But density ρ varies as the square of the above distance. Therefore $\rho = k\left(x^2 + y^2\right)$

In the region, y varies from 0 *to* e^x and x varies from 0 *to* 1.

3 Find the mass of the area bounded by the curves $y = x^2$ *and* $x = y^2$, given that the density of the lamina is $\rho = k\left(x^2 + y^2\right)$.

$$Mass = \int_0^1 \int_{y^2}^{\sqrt{y}} \rho \; dx \; dy$$

$$= k\int_0^1 \int_{y^2}^{\sqrt{y}} x^2 + y^2 \; dx \; dy$$

$$= k\int_0^1 \left[\frac{x^3}{3} + y^2 x\right]_{y^2}^{\sqrt{y}} dy$$

$$= k\int_0^1 \left[\frac{\left(\sqrt{y}\right)^3}{3} + y^2\sqrt{y} - \frac{y^6}{3} - y^4\right] dy$$

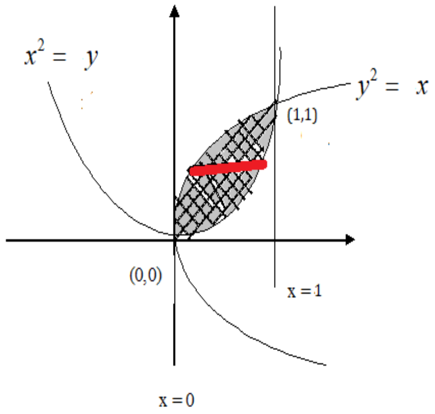
$$= k\int_0^1 \left[\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^6}{3} - y^4\right] dy$$

$$= k\left[\frac{1}{3}\frac{y^{\frac{5}{2}}}{\frac{5}{2}} + \frac{y^{\frac{7}{2}}}{\frac{7}{2}} - \frac{y^7}{3.7} - \frac{y^5}{5}\right]_0^1$$

$$= k\left[\frac{2}{15} + \frac{2}{7} - \frac{1}{21} - \frac{1}{5}\right]$$

$$= \frac{18k}{105}$$

.



In the region, x varies from y^2 *to* $\sqrt{y}$ and y varies from 0 *to* 1.

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4 Find the mass of the plate which is inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$, if the density varies as the distance from the pole.

Therefore required Mass

$$M = \iint_A \rho\ r\ drd\theta$$

$$M = 2\int_0^{\frac{\pi}{3}} \int_a^{2a\cos\theta} \rho\ r\ drd\theta$$

$$= 2k\int_0^{\frac{\pi}{3}} \int_a^{2a\cos\theta} r^2\ dr\ d\theta$$

$$= 2k\int_0^{\frac{\pi}{3}} \left[\frac{r^3}{3}\right]_a^{2a\cos\theta} d\theta$$

$$= \frac{2k}{3}\int_0^{\frac{\pi}{3}} 8a^3\cos^3\theta - a^3\ d\theta$$

$$= \frac{2ka^3}{3}\int_0^{\frac{\pi}{3}} 8\cos^3\theta - 1\ d\theta$$

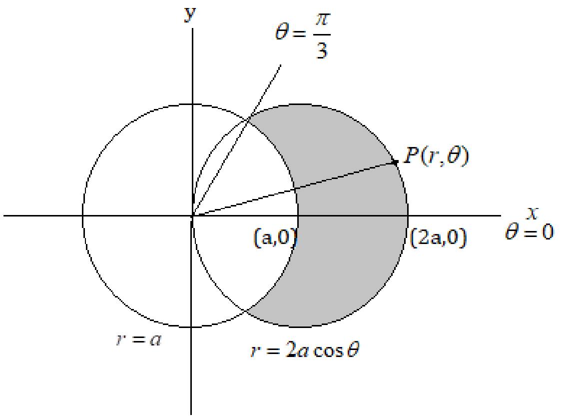
$$= \frac{2ka^3}{3}\int_0^{\frac{\pi}{3}} \frac{8}{4}\cos 3\theta + \frac{3\times 8}{4}\cos\theta - 1\ d\theta,$$

$$\left[\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta\right]$$

$$= \frac{2ka^3}{3}\left[\frac{8}{4}\frac{\sin 3\theta}{3} + \frac{24}{4}\sin\theta - \theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{2ka^3}{3}\left[\frac{24}{4}\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right]$$

$$= \frac{2ka^3}{3}\left[3\sqrt{3} - \frac{\pi}{3}\right]$$



Since the distance of any point $P(r,\theta)$ from the pole is r, the density at that point is given by $\rho = kr$, k is proportionality constant.

Also the shaded portion is symmetry about the initial line $\theta = 0$. Hence the required mass is twice the mass of the area above the initial line.

For the region above the initial line, r varies from a to $2a\cos\theta$ and θ varies from 0 to $\frac{\pi}{3}$.

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5 Find the mass of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, given that the density of the solid is $\rho = kxyz$.

We know that

$$Mass = \int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} \int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} \rho \; dx \; dy \; dz$$

$$= \int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} \int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} k \; xyz \; dx \; dy \; dz$$

$$= k\int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} \left[\frac{x^2}{2}\right]_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} yz \; dy \; dz$$

$$= \frac{k}{2}\int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} a^2\left(\left(1-\frac{z}{c}\right)-\frac{y}{b}\right)^2 yz \; dy \; dz$$

$$= \frac{ka^2}{2}\int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} \left(\left(1-\frac{z}{c}\right)^2+\frac{y^2}{b^2}-2\left(1-\frac{z}{c}\right)\frac{y}{b}\right) yz \; dy \; dz$$

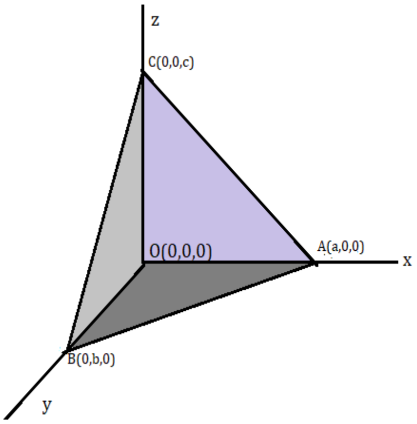
$$= \frac{ka^2}{2}\int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} \left(\left(1-\frac{z}{c}\right)^2 y+\frac{y^3}{b^2}-2\left(1-\frac{z}{c}\right)\frac{y^2}{b}\right) z \; dy \; dz$$

$$= \frac{ka^2}{2}\int_0^c \left(\left(1-\frac{z}{c}\right)^2 \frac{y^2}{2}+\frac{y^4}{4b^2}-2\left(1-\frac{z}{c}\right)\frac{y^3}{3b}\right)_0^{b\left(1-\frac{z}{c}\right)} z \; dz$$

$$= \frac{ka^2}{2}\int_0^c \left[\frac{1}{2}\left(1-\frac{z}{c}\right)^2 b^2\left(1-\frac{z}{c}\right)^2+\frac{1}{4b^2}b^4\left(1-\frac{z}{c}\right)^4-2\left(1-\frac{z}{c}\right)\frac{1}{3b}b^3\left(1-\frac{z}{c}\right)^3\right] z \; dz$$

$$= \frac{ka^2}{2}\int_0^c \left[\frac{b^2}{2}\left(1-\frac{z}{c}\right)^4+\frac{b^2}{4}\left(1-\frac{z}{c}\right)^4-\frac{2b^2}{3}\left(1-\frac{z}{c}\right)^4\right] z \; dz$$

$$= \frac{ka^2}{2}\int_0^c \left[\frac{b^2}{2}+\frac{b^2}{4}-\frac{2b^2}{3}\right]\left(1-\frac{z}{c}\right)^4 z \; dz$$



The tetrahedron $OABC$ is bounded by the planes

$x=0,\ y=0,\ z=0$ & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Here x varies from 0 *to* $a\left(1-\frac{y}{b}-\frac{z}{c}\right)$ and y varies from 0 *to* $b\left(1-\frac{z}{c}\right)$ and z varies from 0 *to* c

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$$= \frac{ka^2b^2}{24}\int_0^c \left(1-\frac{z}{c}\right)^4 z\ dz$$

$$= \frac{ka^2b^2}{24}\int_0^{\frac{\pi}{2}} c\sin^2\theta\left(1-\sin^2\theta\right)^4 2c\sin\theta\cos\theta\ d\theta$$

$$= \frac{ka^2b^2c^2}{12}\int_0^{\frac{\pi}{2}} \sin^3\theta\left(\cos^2\theta\right)^4\cos\theta\ d\theta$$

$$= \frac{ka^2b^2c^2}{12}\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^9\theta\ d\theta$$

$$= \frac{ka^2b^2c^2}{12}\frac{2.\ 8.6.4.2}{12.10.8.6.4.2} = \frac{ka^2b^2c^2}{720}$$

$Put\ \ z = c\sin^2\theta,\ then$

$dz = 2c\sin\theta\cos\theta\ d\theta$

$when\ z = 0,\ \theta = 0$

$when\ z = c,\ c = c\sin^2\theta,$

$i.e.\ \ 1 = \sin\theta,$

$i.e.\ \ \theta = \frac{\pi}{2}$

.

6 Find the mass of the solid bounded by the planes and the planes $y=0, z=0\ and\ z=h$ and the cylinder $x^2+y^2=a^2$, if $\rho = kyz$.

We know that

$$Mass = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^h \rho\ dz\ dy\ dx$$

$$= k\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^h yz\ dz\ dy\ dx$$

$$= k\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} y\left[\frac{z^2}{2}\right]_0^h dy\ dx$$

$$= k\ \frac{h^2}{2}\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} y\,dy\ dx$$

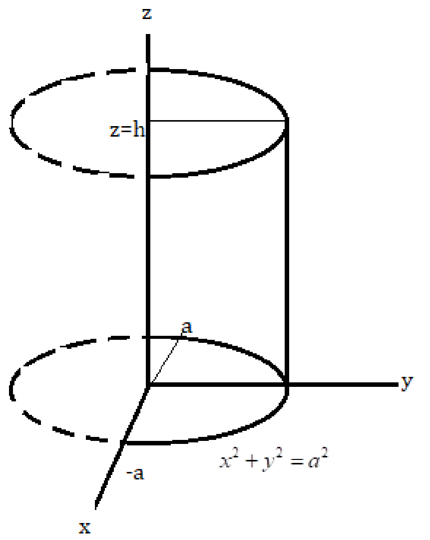
$$= k\ \frac{h^2}{2}\int_{-a}^{a}\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= k\ \frac{h^2}{2}\frac{1}{2}\int_{-a}^{a} a^2-x^2\ dx$$

$$= k\ \frac{h^2}{2}\frac{1}{2}.2\int_0^a a^2-x^2\ dx,\ \because a^2-x^2\ is\ even$$

$$= k\ \frac{h^2}{2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= k\ \frac{h^2}{2}\left[a^3-\frac{a^3}{3}\right] = k\ \frac{h^2a^3}{3}$$



The solid is bounded by the planes $y=0,\ z=0\ and\ z=h$ and the cylinder $x^2+y^2=a^2$.

Here x varies from $-a\ to\ a$

Here z varies from $0\ to\ h$

and y varies from $0\ to\ \sqrt{a^2-x^2}$

(semi cylinder)

Moments and Centers of Gravity of Plane Regions

(i) Let R be a vertically or horizontally simple region. The moment M_x of R about the $x-$axis and the moment M_y of R about the $y-$axis are defined by

$$M_x = \iint_R ydA \quad and \quad M_y = \iint_R xdA$$

(ii) If R has positive area A, then the centre of gravity (centre of mass or centroid) of R is the point $(\overline{x}, \overline{y})$ defined by

$$\overline{x} = \frac{M_y}{A} = \frac{\iint_R xdA}{\iint_R dA} \quad and \quad \overline{y} = \frac{M_x}{A} = \frac{\iint_R ydA}{\iint_R dA}$$

(iii) The centre of gravity of a lamina of area A in the $xy-$plane and having density $\rho = f(x, y)$ is given by

$$\overline{x} = \frac{\iint_A x\rho \, dxdy}{\iint_A \rho \, dxdy} \quad and \quad \overline{y} = \frac{\iint_A y\rho \, dxdy}{\iint_A \rho \, dxdy}$$

In polar coordinates $\overline{x} = \dfrac{\iint_A r\cos\theta \; \rho \; r \; drd\theta}{\iint_A \rho \; r \; drd\theta}$ and $\overline{y} = \dfrac{\iint_A r\sin\theta \; \rho \; r \; drd\theta}{\iint_A \rho \; r \; drd\theta}$

(iv) The centre of gravity of a solid of volume V having the density $\rho = f(x, y, z)$ is given by

$$\overline{x} = \frac{\iiint_V x\rho \, dxdydz}{\iiint_V \rho \, dxdydz}, \quad \overline{y} = \frac{\iiint_V y\rho \, dxdydz}{\iiint_V \rho \, dxdydz} \quad and \quad \overline{z} = \frac{\iiint_V z\rho \, dxdydz}{\iiint_V \rho \, dxdydz}$$

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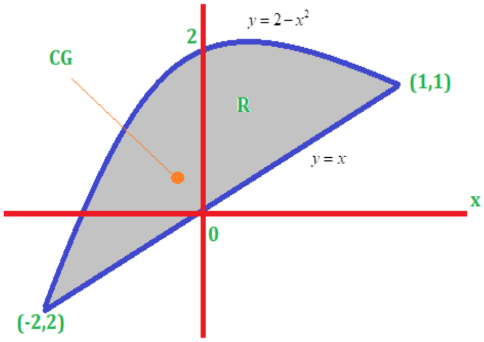
1 Let R be the plane region bounded by the line $y = x$ and the parabola $y = 2 - x^2$. Find the moments of R about the x and y axes and determine the centre of gravity of R.

Solving $y = 2 - x^2$ and $y = x$, we get the points of intersection.

$$x = 2 - x^2$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2, 1$$
$$\therefore\ y = -2, 1$$



$$M_x = \iint_R y\,dA$$
$$= \int_{-2}^{1}\int_{x}^{2-x^2} y\ dy\ dx$$
$$= \int_{-2}^{1}\left[\frac{y^2}{2}\right]_{x}^{2-x^2} dx$$
$$= \frac{1}{2}\int_{-2}^{1} \left(2-x^2\right)^2 - x^2\ dx$$
$$= \frac{1}{2}\int_{-2}^{1} 4 + x^4 - 4x^2\ dx$$
$$= \frac{1}{2}\left[4x + \frac{x^5}{5} - 4\frac{x^3}{3}\right]_{-2}^{1}$$
$$= \frac{1}{2}\left[1 + \frac{1}{5} - \frac{4}{3} + 8 + \frac{32}{5} - \frac{32}{3}\right]$$
$$= \frac{9}{5}$$

$$M_y = \iint_R x\,dA$$
$$= \int_{-2}^{1}\int_{x}^{2-x^2} x\ dy\ dx$$
$$= \int_{-2}^{1} x\left(2 - x^2 - x\right)\ dx$$
$$= \int_{-2}^{1} 2x - x^3 - x^2\ dx$$
$$= \left[\frac{2x^2}{2} - \frac{x^4}{4} - \frac{x^3}{3}\right]_{-2}^{1}$$
$$= \left[1 - \frac{1}{4} - \frac{1}{3} - 4 + 4 - \frac{8}{3}\right]$$
$$= -\frac{9}{4}$$

The area of R is given by

$$A = \iint_R dA = \int_{-2}^{1}\int_{x}^{2-x^2} dy\ dx$$
$$= \int_{-2}^{1} 2 - x^2 - x\ dx = \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1}$$
$$= \left[2 - \frac{1}{3} - \frac{1}{2} + 4 - \frac{8}{3} + 2\right]$$
$$= \left[8 - \frac{1}{3} - \frac{1}{2} - \frac{8}{3}\right]$$
$$= \frac{9}{2}$$

Hence

$$\bar{x} = \frac{M_y}{A} = -\frac{9}{4}\times\frac{2}{9} = -\frac{1}{2} \quad and$$

$$\bar{y} = \frac{M_x}{A} = \frac{9}{5}\times\frac{2}{9} = \frac{2}{5}$$

Therefore the centre of gravity of R is the point $\left(-\frac{1}{2}, \frac{2}{5}\right)$

2 Find the centre of gravity of a lamina in the form of a quadrant of asteroid $\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}}+\frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}}=1$, given that the density of the lamina is $\rho = kxy$.

The point of centre of gravity is given by

$$\overline{x}=\frac{\iint\limits_A x\rho\ dxdy}{\iint\limits_A \rho\ dxdy} \quad \text{and} \quad \overline{y}=\frac{\iint\limits_A y\rho\ dxdy}{\iint\limits_A \rho\ dxdy}$$

$$\iint\limits_A \rho\ dxdy = k\int_0^a\int_0^{y_1} xy\ dydx$$

$$=k\int_0^a x\left[\frac{y^2}{2}\right]_0^{y_1} dx$$

$$=\frac{k}{2}\int_0^a xy_1^2\ dx$$

$$=\frac{k}{2}\int_0^a x.b^2\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^3 dx$$

$$=\frac{kb^2}{2}\int_0^{\frac{\pi}{2}} a\sin^3\theta.\left[1-\left(\sin^3\theta\right)^{\frac{2}{3}}\right]^3 3a\sin^2\theta\cos\theta\ d\theta$$

$$=\frac{3ka^2b^2}{2}\int_0^{\frac{\pi}{2}} \sin^5\theta.\left[1-\sin^2\theta\right]^3 \cos\theta\ d\theta$$

$$=\frac{3ka^2b^2}{2}\int_0^{\frac{\pi}{2}} \sin^5\theta.\cos^7\theta\ d\theta$$

$$=\frac{3ka^2b^2}{2}.\frac{4.2\ \ 6.4.2}{12.10.8.6.4.2}$$

$$=\frac{1}{80}ka^2b^2$$

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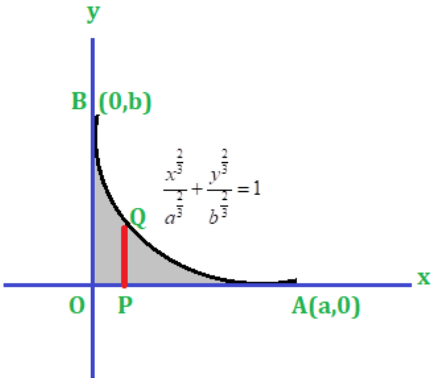

From the given curve

$$\frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}}=1-\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} \qquad \frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}}=1-\frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}}$$

$$y^{\frac{2}{3}}=b^{\frac{2}{3}}\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right] \qquad x^{\frac{2}{3}}=a^{\frac{2}{3}}\left[1-\left(\frac{y}{b}\right)^{\frac{2}{3}}\right]$$

$$y=b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}=y_1 \qquad x=a\left[1-\left(\frac{y}{b}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}=x_1$$

For the area $OABO$,

x varies from $0\ to\ a$ $\qquad$ y varies from $0\ to\ b$

y varies from $0\ to\ y_1$ $\qquad$ x varies from $0\ to\ x_1$

Put

$x=a\sin^3\theta,\ then\ \ dx=3a\sin^2\theta\cos\theta\ d\theta$

$when\ \ x=0,\ \theta=0$

$when\ \ x=a,\ a=a\sin^3\theta,\ i.e.\ \ 1=\sin\theta,\ i.e.\ \ \theta=\frac{\pi}{2}$

If

$y=b\cos^3\theta,\ then\ \ dx=-3b\cos^2\theta\sin\theta\ d\theta$

$when\ \ y=0,\ \theta=\frac{\pi}{2}$

$when\ \ y=b,\ b=b\cos^3\theta,\ i.e.\ \ 1=\cos\theta,\ i.e.\ \ \theta=0$

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$$\iint_A x\rho \ dxdy$$

$$= k\int_0^a \int_0^{y_1} x^2 y \ dydx$$

$$= k\int_0^a x^2 \left[\frac{y^2}{2}\right]_0^{y_1} dx$$

$$= \frac{k}{2}\int_0^a x^2 y_1^2 \ dx$$

$$= \frac{k}{2}\int_0^a x^2 .b^2 \left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^3 dx$$

$$= \frac{kb^2}{2}\int_0^{\frac{\pi}{2}} a^2 \sin^6\theta.\left[1-\left(\sin^3\theta\right)^{\frac{2}{3}}\right]^3 3a\sin^2\theta\cos\theta \ d\theta$$

$$= \frac{3ka^3b^2}{2}\int_0^{\frac{\pi}{2}} \sin^8\theta.\left[1-\sin^2\theta\right]^3 \cos\theta \ d\theta$$

$$= \frac{3ka^3b^2}{2}\int_0^{\frac{\pi}{2}} \sin^8\theta.\cos^7\theta \ d\theta$$

$$= \frac{3ka^3b^2}{2}.\frac{7.5.3.1 \ \ 6.4.2}{15.13.11.9.7.5.3.1}$$

$$= \frac{8}{2145}ka^3b^2$$

Hence $\bar{x} = \frac{8ka^3b^2}{2145}\times\frac{80}{ka^2b^2} = \frac{128a}{429}$

$$\iint_A y\rho \ dxdy$$

$$= k\int_0^b \int_0^{x_1} xy^2 \ dxdy$$

$$= k\int_0^b y^2 \left[\frac{x^2}{2}\right]_0^{x_1} dy$$

$$= \frac{k}{2}\int_0^b y^2 x_1^2 \ dy$$

$$= \frac{k}{2}\int_0^b y^2 .a^2 \left[1-\left(\frac{y}{b}\right)^{\frac{2}{3}}\right]^3 dy$$

$$= -\frac{ka^2}{2}\int_{\frac{\pi}{2}}^0 b^2 \cos^6\theta.\left[1-\left(\cos^3\theta\right)^{\frac{2}{3}}\right]^3 3b\cos^2\theta\sin\theta \ d\theta$$

$$= \frac{3ka^2b^3}{2}\int_{\frac{\pi}{2}}^0 \cos^8\theta.\left[1-\cos^2\theta\right]^3 \sin\theta \ d\theta$$

$$= \frac{3ka^2b^3}{2}\int_0^{\frac{\pi}{2}} \cos^8\theta.\sin^7\theta \ d\theta$$

$$= \frac{3ka^2b^3}{2}.\frac{7.5.3.1 \ \ 6.4.2}{15.13.11.9.7.5.3.1}$$

$$= \frac{8}{2145}ka^2b^3$$

Hence $\bar{y} = \frac{8ka^2b^3}{2145}\times\frac{80}{ka^2b^2} = \frac{128b}{429}$

3 Find by double integration, the centre of gravity of the area of the circle $x^2+y^2=a^2$ lying in the first quadrant.

The point of centre of gravity is given by

$$\bar{x}=\frac{\iint\limits_A x\rho\ dxdy}{\iint\limits_A \rho\ dxdy} \text{ and } \bar{y}=\frac{\iint\limits_A y\rho\ dxdy}{\iint\limits_A \rho\ dxdy}$$

Since the density of the lamina is not given, it can be taken as $\rho=k,$ a constant.

$$\iint\limits_A \rho\ dxdy=k\int_0^a\int_0^{\sqrt{a^2-x^2}} dydx$$

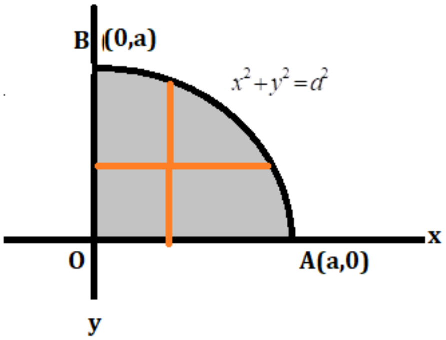
$$=k\int_0^a \sqrt{a^2-x^2}\ dx$$

$$=k\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=k\left[\frac{a^2}{2}\sin^{-1}1\right], \because \sin^{-1}0=0$$

$$=k\left[\frac{a^2}{2}.\frac{\pi}{2}\right], \quad \because \sin\frac{\pi}{2}=1$$

$$=\frac{k\pi a^2}{4}$$



From the given curve

$y^2=a^2-x^2,\ i.e.\ y=\sqrt{a^2-x^2}$

For the area $OABO$,

x varies from $0\ to\ a$ & y varies from $0\ to\ \sqrt{a^2-x^2}$

Also, from the given curve

$x^2=a^2-y^2,\ i.e.\ x=\sqrt{a^2-y^2}$

For the area $OABO$,

y varies from $0\ to\ a$ & x varies from $0\ to\ \sqrt{a^2-y^2}$

$$\iint\limits_A x\rho\ dxdy=k\int_0^a\int_0^{\sqrt{a^2-y^2}} x\ dx\,dy$$

$$=k\int_0^a\left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$=\frac{k}{2}\int_0^a a^2-y^2\ dy$$

$$=\frac{k}{2}\left[a^2y-\frac{y^3}{3}\right]_0^a$$

$$=\frac{k}{2}\left[a^3-\frac{a^3}{3}\right]$$

$$=\frac{ka^3}{3}$$

Hence $\bar{x}=\frac{ka^3}{3}\times\frac{4}{k\pi a^2}=\frac{4a}{3\pi}$

$$\iint\limits_A y\rho\ dydx=k\int_0^a\int_0^{\sqrt{a^2-x^2}} y\ dy\,dx$$

$$=k\int_0^a\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$=\frac{k}{2}\int_0^a a^2-x^2\ dx$$

$$=\frac{k}{2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$=\frac{k}{2}\left[a^3-\frac{a^3}{3}\right]$$

$$=\frac{ka^3}{3}$$

Hence $\bar{y}=\frac{ka^3}{3}\times\frac{4}{k\pi a^2}=\frac{4a}{3\pi}$

https://doi.org/10.5281/zenodo.15288481

4 A plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes. Find the coordinates of the centroid.

The point of centre of gravity is given by

$$\overline{x}=\frac{\iint\limits_A x\rho\ dxdy}{\iint\limits_A \rho\ dxdy} \text{ and } \overline{y}=\frac{\iint\limits_A y\rho\ dxdy}{\iint\limits_A \rho\ dxdy}$$

Since the density of the lamina is proportional to the product of the distances of the point from the axes, it can be taken as $\rho=kxy$, k is a constant.

$$\iint\limits_A \rho\ dxdy=k\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\ dxdy$$

$$=k\int_0^b y\left[\frac{x^2}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$=\frac{k}{2}\int_0^b y\ \frac{a^2}{b^2}\left(b^2-y^2\right) dy$$

$$=\frac{k}{2}\frac{a^2}{b^2}\int_0^b \left(b^2y-y^3\right) dy$$

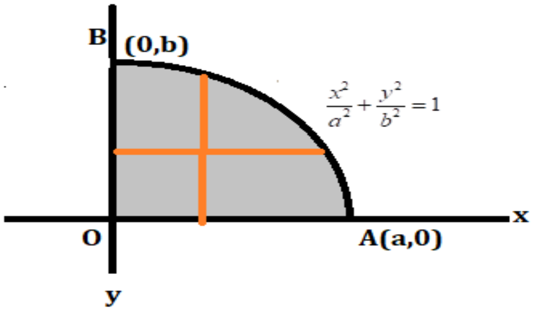
$$=\frac{k}{2}\frac{a^2}{b^2}\left[\frac{b^2y^2}{2}-\frac{y^4}{4}\right]_0^b$$

$$=\frac{k}{2}\frac{a^2}{b^2}\left[\frac{b^4}{2}-\frac{b^4}{4}\right]$$

$$=\frac{k}{2}\frac{a^2}{b^2}\frac{b^4}{4}$$

$$=\frac{ka^2b^2}{8}$$

.



From the given curve

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2},\ \ i.e.\ \ \frac{y^2}{b^2}=\frac{a^2-x^2}{a^2}$$

$$y^2=b^2\frac{a^2-x^2}{a^2},\ \ i.e.\ \ y=\frac{b}{a}\sqrt{a^2-x^2}$$

For the area $OABO$,
x varies from $0\ to\ a$ & y varies from $0\ to\ \frac{b}{a}\sqrt{a^2-x^2}$

https://doi.org/10.5281/zenodo.15288481

$$\iint_A y\rho\ dydx = k\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy^2\ dy\,dx$$

$$= k\int_0^a x\left[\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$=\frac{k}{3}\int_0^a x\frac{b^3}{a^3}\left(a^2-x^2\right)^{\frac{3}{2}}\ dx$$

$$=\frac{k}{3}\frac{b^3}{a^3}\frac{1}{2}\int_0^a \left(a^2-x^2\right)^{\frac{3}{2}}\ d\left(x^2\right)$$

$$=\frac{k}{3}\frac{b^3}{a^3}\frac{1}{2}\left[\frac{\left(a^2-x^2\right)^{\frac{5}{2}}}{-\frac{5}{2}}\right]_0^a$$

$$=-\frac{k}{3}\frac{b^3}{a^3}\frac{1}{2}\frac{2}{5}\left[0-a^5\right]$$

$$=\frac{k}{3}\frac{b^3}{a^3}\frac{1}{2}\frac{2}{5}a^5$$

$$=\frac{k}{2}.\frac{b^2}{a^2}.\frac{2a^5}{15}$$

$$=\frac{ka^2b^3}{15}$$

Hence $\bar{y}=\frac{ka^2b^3}{15}\times\frac{8}{ka^2b^2}=\frac{8b}{15}$

.

$$\iint_A x\rho\ dydx = k\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2y\ dy\,dx$$

$$= k\int_0^a x^2\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$=\frac{k}{2}\int_0^a x^2\frac{b^2}{a^2}\left(a^2-x^2\right) dx$$

$$=\frac{k}{2}.\frac{b^2}{a^2}\int_0^a \left(a^2x^2-x^4\right) dx$$

$$=\frac{k}{2}.\frac{b^2}{a^2}\left[a^2\frac{x^3}{3}-\frac{x^5}{5}\right]_0^a$$

$$=\frac{k}{2}.\frac{b^2}{a^2}\left[\frac{a^5}{3}-\frac{a^5}{5}\right]$$

$$=\frac{k}{2}.\frac{b^2}{a^2}.\frac{2a^5}{15}$$

$$=\frac{ka^3b^2}{15}$$

Hence $\bar{x}=\frac{ka^3b^2}{15}\times\frac{8}{ka^2b^2}=\frac{8a}{15}$

5 Find by double integration, the centre of gravity of the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ lying in the first quadrant.

The point of centre of gravity is given by

$$\overline{x}=\frac{\iint_A x\rho\ dxdy}{\iint_A \rho\ dxdy} \quad \text{and} \quad \overline{y}=\frac{\iint_A y\rho\ dxdy}{\iint_A \rho\ dxdy}$$

Since the density of the lamina is not given, it can be taken as $\rho=k,$ a constant.

$$\iint_A \rho\ dxdy = k\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dxdy$$

$$=k\int_0^b \frac{a}{b}\sqrt{b^2-y^2}\ dy$$

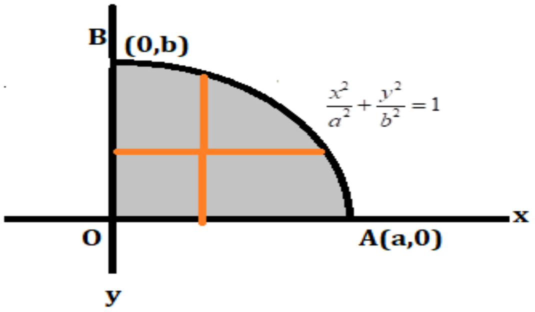
$$=k\frac{a}{b}\left[\frac{y}{2}\sqrt{b^2-y^2}+\frac{b^2}{2}\sin^{-1}\frac{y}{b}\right]_0^b$$

$$=k\frac{a}{b}\left[\frac{b^2}{2}\sin^{-1}1\right], \because \sin^{-1}0=0$$

$$=k\frac{a}{b}\left[\frac{b^2}{2}.\frac{\pi}{2}\right], \quad \because \ \sin\frac{\pi}{2}=1$$

$$=\frac{k\pi ab}{4}$$

.



From the given curve

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}, \ i.e.\ \frac{y^2}{b^2}=\frac{a^2-x^2}{a^2}$$

$$y^2=b^2\frac{a^2-x^2}{a^2}, \ i.e.\ y=\frac{b}{a}\sqrt{a^2-x^2}$$

For the area $OABO$,
x varies from $0\ to\ a$ & y varies from $0\ to\ \frac{b}{a}\sqrt{a^2-x^2}$

Also, from the given curve

$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}, \ i.e.\ \frac{x^2}{a^2}=\frac{b^2-y^2}{b^2}$$

$$x^2=a^2\frac{b^2-y^2}{b^2}, \ i.e.\ x=\frac{a}{b}\sqrt{b^2-y^2}$$

For the area $OABO$,
y varies from $0\ to\ b$ & x varies from $0\ to\ \frac{a}{b}\sqrt{b^2-y^2}$

$$\iint_A x\rho\; dxdy = k\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} x\; dx\,dy$$

$$= k\int_0^b \left[\frac{x^2}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \frac{k}{2}\int_0^b \frac{a^2}{b^2}\left(b^2-y^2\right)\, dy$$

$$= \frac{k}{2}\cdot\frac{a^2}{b^2}\left[b^2y-\frac{y^3}{3}\right]_0^b$$

$$= \frac{k}{2}\cdot\frac{a^2}{b^2}\left[b^3-\frac{b^3}{3}\right]$$

$$= \frac{kb^3}{2}\cdot\frac{a^2}{b^2}\cdot\frac{2}{3}$$

$$= \frac{ka^2b}{3}$$

Hence $\bar{x} = \frac{ka^2b}{3}\times\frac{4}{k\pi ab} = \frac{4a}{3\pi}$

.

$$\iint_A y\rho\; dydx = k\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\; dy\,dx$$

$$= k\int_0^a \left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{k}{2}\int_0^a \frac{b^2}{a^2}\left(a^2-x^2\right)\, dx$$

$$= \frac{k}{2}\cdot\frac{b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

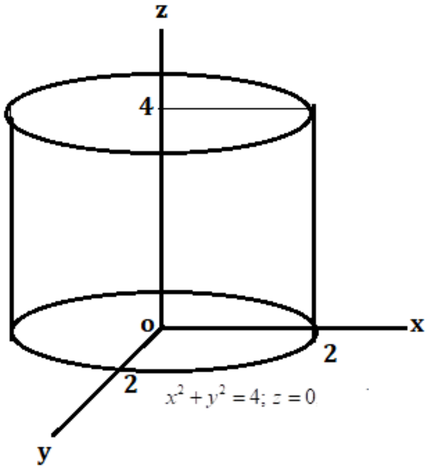
$$= \frac{k}{2}\cdot\frac{b^2}{a^2}\left[a^3-\frac{a^3}{3}\right]$$

$$= \frac{ka^3}{2}\cdot\frac{b^2}{a^2}\cdot\frac{2}{3}$$

$$= \frac{kab^2}{3}$$

Hence $\bar{y} = \frac{kab^2}{3}\times\frac{4}{k\pi ab} = \frac{4b}{3\pi}$

6 Find the centre of gravity of a solid of volume V bounded by the circular cylinder $x^2+y^2=4;\ z=0,\ z=4$ whose density is given by $\rho = 20 - z^2$.



The centre of gravity of a solid is given by

$$\bar{x} = \frac{\iiint_V x\rho\; dxdydz}{\iiint_V \rho\; dxdydz},\quad \bar{y} = \frac{\iiint_V y\rho\; dxdydz}{\iiint_V \rho\; dxdydz}\quad and \quad \bar{z} = \frac{\iiint_V z\rho\; dxdydz}{\iiint_V \rho\; dxdydz}$$

Let us use cylindrical coordinates to evaluate certain integrals:

Changing to cylindrical polar co-ordinates by the relations $x = r\cos\theta,\ y = r\sin\theta\ and\ z = z$ the region $\left\{x^2+y^2 \le 4;\ 0 \le z \le 4\right\}$ is transformed to the region $\left\{(r,\theta,z): 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 4\right\}$. Here $dxdydz$ is to be replaced by $r\ dr\ d\theta\ dz$.

$$\iiint_V \rho\ dxdydz$$

$$= \int_0^{2\pi}\int_0^2\int_0^4 20-z^2\, r\ dz\ dr\ d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\left[20z-\frac{z^3}{3}\right]_0^4 dr\ d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\left[80-\frac{64}{3}\right] dr\ d\theta$$

$$= \frac{176}{3}\int_0^{2\pi}\int_0^2 r\ dr\ d\theta$$

$$= \frac{176}{3}\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^2 d\theta$$

$$= \frac{176}{3}\times 2\int_0^{2\pi} d\theta$$

$$= \frac{176}{3}\times 2\left[2\pi-0\right]$$

$$= \frac{704}{3}\pi$$

$$\iiint_V \rho z\ dxdydz$$

$$= \int_0^{2\pi}\int_0^2\int_0^4 \left(20-z^2\right)z\ r\ dz\ dr\ d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\left[\frac{20z^2}{2}-\frac{z^4}{4}\right]_0^4 dr\ d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\left[160-64\right] dr\ d\theta$$

$$= 96\int_0^{2\pi}\int_0^2 r\ dr\ d\theta$$

$$= 96\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^2 d\theta$$

$$= 96\times 2\int_0^{2\pi} d\theta$$

$$= 96\times 2\left[2\pi-0\right]$$

$$= 384\pi$$

$$\iiint_V \rho x\ dxdydz$$

$$= \int_0^4\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-2}^{2}\left(20-z^2\right)x\ dx\ dy\ dz$$

$= 0$, since the integrand x is odd

Also

$$\iiint_V \rho y\ dxdydz$$

$$= \int_0^4\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{-2}^{2}\left(20-z^2\right)y\ dy\ dx\ dz$$

$= 0$, since the integrand y is odd

$$\bar{x} = \frac{\iiint_V x\rho\ dxdydz}{\iiint_V \rho\ dxdydz} = 0 \qquad \bar{y} = \frac{\iiint_V y\rho\ dxdydz}{\iiint_V \rho\ dxdydz} = 0 \qquad \bar{z} = \frac{\iiint_V x\rho\ dxdydz}{\iiint_V \rho\ dxdydz} = \frac{384\pi}{704\pi}\times 3 = \frac{18}{11}$$

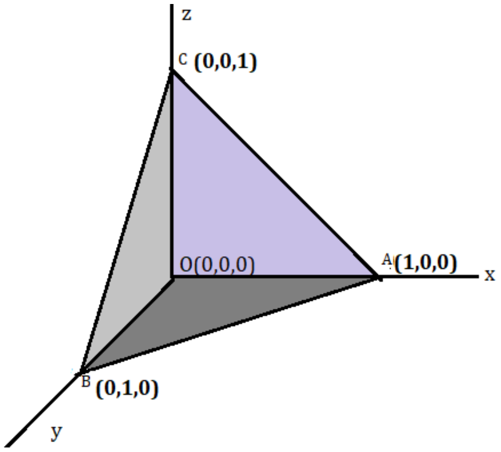
.

7 Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$, given that the density at any point varying as its distance from the face $z=0$.

We know that the distance from the face $z=0$ to any point in the tetradedron is $\rho = kz$, k is a constant.

The centre of gravity of a solid is given by

$$\bar{x} = \frac{\iiint_V x\rho\ dxdydz}{\iiint_V \rho\ dxdydz},$$



The tetrahedron $OABC$ is bounded by the planes $x=0,\ y=0,\ z=0\ \&\ x+y+z=1$

Here x varies from $0\ to\ 1-y-z$ and y varies from $0\ to\ 1-z$ and z varies from $0\ to\ 1$

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$$\overline{y} = \frac{\iiint\limits_V y\rho \; dxdydz}{\iiint\limits_V \rho \; dxdydz}$$

$$and \quad \overline{z} = \frac{\iiint\limits_V z\rho \; dxdydz}{\iiint\limits_V \rho \; dxdydz}$$

$$\iiint\limits_V \rho \; dxdydz = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} \rho \; dx \; dy \; dz$$

$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} k \; z \; dx \; dy \; dz$$

$$= k \int_0^1 \int_0^{1-z} \left(1 - y - z\right) z \; dy \; dz$$

$$= k \int_0^1 \int_0^{1-z} \left(z - zy - z^2\right) dy \; dz$$

$$= k \int_0^1 \left(zy - \frac{y^2 z}{2} - z^2 y\right)_0^{1-z} dz$$

$$= k \int_0^1 \left(z(1-z) - \frac{(1-z)^2 z}{2} - z^2(1-z)\right) dz$$

$$= k \int_0^1 \left(z - z^2 - \frac{z}{2} - \frac{z^3}{2} + z^2 - z^2 + z^3\right) dz$$

$$= k \int_0^1 \left(\frac{z}{2} - z^2 + \frac{z^3}{2}\right) dz$$

$$= k \left(\frac{z^2}{4} - \frac{z^3}{3} + \frac{z^4}{8}\right)_0^1$$

$$= k \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right)$$

$$= \frac{k}{24}$$

$$\iiint\limits_V \rho x \; dxdydz = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} \rho x \; dx \; dy \; dz$$

$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} k \; xz \; dx \; dy \; dz$$

$$= k \int_0^1 \int_0^{1-z} z \left[\frac{x^2}{2}\right]_0^{1-y-z} dy \; dz$$

$$= \frac{k}{2} \int_0^1 \int_0^{1-z} z \left((1-z) - y\right)^2 \; dy \; dz$$

$$= \frac{k}{2} \int_0^1 z \left(\frac{\left((1-z) - y\right)^3}{-3}\right)_0^{1-z} dz$$

$$= -\frac{k}{2 \times 3} \int_0^1 z \left[0 - (1-z)^3\right] dz$$

$$= \frac{k}{6} \int_0^1 z(1-z)^3 \; dz$$

$$= \frac{k}{6} \int_0^1 z\left(1 - z^3 + 3z^2 - 3z\right) dz$$

$$= \frac{k}{6} \int_0^1 \left(z - z^4 + 3z^3 - 3z^2\right) dz$$

$$= \frac{k}{6} \left(\frac{z^2}{2} - \frac{z^5}{5} + \frac{3z^4}{4} - \frac{3z^3}{3}\right)_0^1$$

$$= \frac{k}{6} \left(\frac{1}{2} - \frac{1}{5} + \frac{3}{4} - \frac{3}{3}\right)$$

$$= \frac{k}{120}$$

$$\iiint_V \rho z\ dxdydz = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} \rho z\ dx\ dy\ dz$$

$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} k\ z^2\ dx\ dy\ dz$$

$$= k \int_0^1 \int_0^{1-z} \left[\frac{z^3}{3}\right]_0^{1-y-z} dy\ dz$$

$$= \frac{k}{3} \int_0^1 \int_0^{1-z} \left((1-z)-y\right)^3\ dy\ dz$$

$$= \frac{k}{3} \int_0^1 \left(\frac{\left[(1-z)-y\right]^4}{-4}\right)_0^{1-z} dz$$

$$= \frac{k}{-4\times 3} \int_0^1 0-\left(1-z\right)^4 dz$$

$$= \frac{k}{12}\left[\frac{\left(1-z\right)^5}{-5}\right]_0^1$$

$$= \frac{k}{12}\left(0+\frac{1}{5}\right)$$

$$= \frac{k}{60}$$

$$\iiint_V \rho y\ dxdydz = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} \rho y\ dx\ dy\ dz$$

$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} k\ yz\ dx\ dy\ dz$$

$$= k \int_0^1 \int_0^{1-z} yz(1-y-z)\ dy\ dz$$

$$= k \int_0^1 \int_0^{1-z} \left(yz - y^2 z - yz^2\right) dy\ dz$$

$$= k \int_0^1 \left(\frac{y^2 z}{2} - \frac{y^3 z}{3} - \frac{y^2 z^2}{2}\right)_0^{1-z} dz$$

$$= k \int_0^1 \frac{z}{2}(1-z)^2 - \frac{z}{3}(1-z)^3 - \frac{z^2}{2}(1-z)^2 dz$$

$$= k \int_0^1 \frac{1}{2}\left(z + z^3 - 2z^2\right) - \frac{1}{3}\left(z - z^4 + 3z^3 - 3z^2\right)$$

$$- \frac{1}{2}\left(z^2 + z^4 - 2z^3\right) dz$$

$$= k\left[\frac{1}{2}\left(\frac{z^2}{2} + \frac{z^4}{4} - \frac{2z^3}{3}\right)_0^1 - \frac{1}{3}\left(\frac{z^2}{2} - \frac{z^5}{5} + \frac{3z^4}{4} - \frac{3z^3}{3}\right)_0^1\right]$$

$$- k\frac{1}{2}\left(\frac{z^3}{3} + \frac{z^5}{5} - \frac{2z^4}{4}\right)_0^1$$

$$= k\left[\frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right) - \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5} + \frac{3}{4} - \frac{3}{3}\right)\right]$$

$$- k\frac{1}{2}\left(\frac{1}{3} + \frac{1}{5} - \frac{2}{4}\right)$$

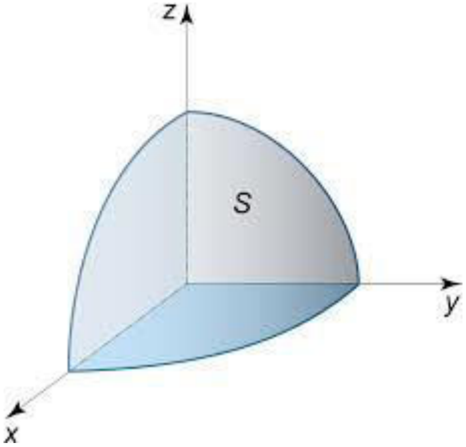
$$= k\left[\frac{1}{24} - \frac{1}{60} - \frac{1}{60}\right]$$

$$= \frac{k}{120}$$

$$\overline{x} = \frac{k}{120}\times\frac{24}{k} = \frac{1}{5}, \qquad \overline{y} = \frac{k}{120}\times\frac{24}{k} = \frac{1}{5}, \qquad \overline{z} = \frac{k}{60}\times\frac{24}{k} = \frac{2}{5}$$

.

8 Find the centre of gravity of the positive octant of the sphere of volume V given by $x^2+y^2+z^2=a^2$ whose density is given by $\rho=kxyz$.



The centre of gravity of a solid is given by

$$\bar{x}=\frac{\iiint\limits_V x\rho\ dxdydz}{\iiint\limits_V \rho\ dxdydz},\quad \bar{y}=\frac{\iiint\limits_V y\rho\ dxdydz}{\iiint\limits_V \rho\ dxdydz}\quad and\quad \bar{z}=\frac{\iiint\limits_V z\rho\ dxdydz}{\iiint\limits_V \rho\ dxdydz}$$

Let us use spherical coordinates to evaluate the integrals:

Changing to spherical polar co-ordinates by the relations $x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi\ and\ z=r\cos\theta$ the region $x^2+y^2+z^2=a^2$ is transformed to the region $\left\{(r,\theta,\phi):0\le r\le a,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\phi\le\frac{\pi}{2}\right\}$. Here $dxdydz$ is to be replaced by $r^2\sin\theta\ dr\ d\theta\ d\phi$.

$$\iiint\limits_V x\rho\ dxdydz=k\int\int\limits_V\int x^2yz\ dxdydz$$

$$=k\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^a r^2\sin^2\theta\cos^2\phi.r\sin\theta\sin\phi.r\cos\theta\ r^2\sin\theta\ dr\ d\theta\ d\phi$$

$$=k\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^a \sin^4\theta\cos\theta\sin\phi\cos^2\phi\ r^6\ dr\ d\theta\ d\phi$$

$$=k\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\sin\phi\cos^2\phi\left[\frac{r^7}{7}\right]_0^a d\theta\ d\phi$$

$$=k\frac{a^7}{7}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\sin^4\theta\cos\theta\right)\sin\phi\cos^2\phi\ \ d\theta\ d\phi$$

$$=k\frac{a^7}{7}\int_0^{\frac{\pi}{2}}\frac{3.1}{5.3.1}\sin\phi\cos^2\phi\ \ d\phi$$

$$=k\frac{a^7}{7}\frac{1}{5}\int_0^{\frac{\pi}{2}}\sin\phi\cos^2\phi\ \ d\phi$$

$$=k\frac{a^7}{7}\frac{1}{5}\frac{1}{3.1}$$

$$=\frac{ka^7}{105}$$

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$$\iiint_V \rho \; dxdydz = k \int \int_V \int xyz \; dxdydz$$

$$= k \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{a} r\sin\theta\cos\phi . r\sin\theta\sin\phi . r\cos\theta \; r^2 \sin\theta \; dr \; d\theta \; d\phi$$

$$= k \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{a} \sin^3\theta\cos\theta . \sin\phi\cos\phi \; r^5 \; dr \; d\theta \; d\phi$$

$$= k \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta . \sin\phi\cos\phi \left[\frac{r^6}{6}\right]_0^a \; d\theta \; d\phi$$

$$= k\frac{a^6}{6} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(\sin^3\theta\cos\theta\right) . \sin\phi\cos\phi \;\; d\theta \; d\phi$$

$$= k\frac{a^6}{6} \int_0^{\frac{\pi}{2}} \frac{2}{3.1} \sin\phi\cos\phi \;\; d\phi$$

$$= k\frac{a^6}{6}\frac{2}{3}\frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2\phi \;\; d\phi, \;\; \because \sin 2x = 2\sin x\cos x$$

$$= k\frac{a^6}{6}\frac{2}{3}\frac{1}{2} \left[-\frac{\cos 2\phi}{2}\right]_0^{\frac{\pi}{2}}$$

$$= k\frac{a^6}{6}\frac{2}{3}\frac{1}{2} \left[\frac{1}{2}+\frac{1}{2}\right], \quad \because \cos\pi = -1, \; \cos 0 = 1$$

$$= \frac{ka^6}{18}$$

$$\overline{x} = \frac{\iiint_V x\rho \; dxdydz}{\iiint_V \rho \; dxdydz} = \frac{ka^7}{105} \times \frac{18}{ka^6} = \frac{6a}{35}$$ By symmetry of the solid w.r.t the axes, $\overline{y} = \overline{z} = \frac{6a}{35}$

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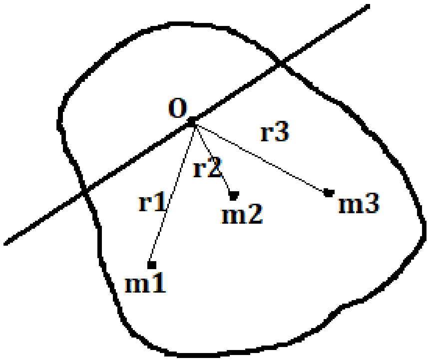
Moment of Inertia

A rigid body rotating about an axis has always a tendency to oppose its state of rotation exactly in the same way as the mass of a particle oppose the tendency to its state of translatory motion. This property of a rotating body is called its Moment of Inertia.



A particle of mass m situated at a distance r from a given axis, the product mr^2 is called the moment of inertia of the particle about the given axis.

In case of a rigid body where there is a continuous distribution of matter, the moment of inertia about a given axis is obtained by integration.

Moment of Inertia of Plane Lamina: Let A be the area of a plane lamina and ρ its density. Then the moment of inertia of the lamina about $x-$axis is given by $I_x = \iint_A \rho y^2 \, dxdy$.

Similarly, the moment of inertia of the lamina about $y-$axis is given by $I_y = \iint_A \rho x^2 \, dxdy$.

Moment of Inertia of a Solid: Let V be a volume of a solid and ρ its density. Let $P(x, y, z)$ be a point of the solid and its distance from $x-$axis is $\sqrt{y^2 + z^2}$. Then the moment of inertia of the solid about the $x-$axis is given by $I_x = \iiint_V \rho\left(y^2 + z^2\right) dxdydz$. Similarly the moment of inertia of the solid about $y-$axis and $z-$axis is

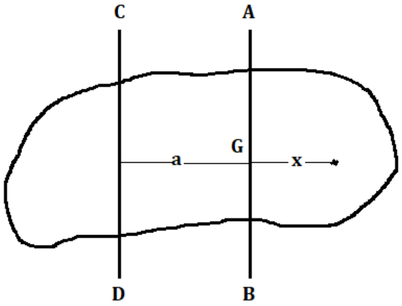
$I_y = \iiint_V \rho\left(x^2 + z^2\right) dxdydz \text{ and } I_z = \iiint_V \rho\left(x^2 + y^2\right) dxdydz$ respectively.

To evaluate the moment of inertia about a line other than coordinate axes, the following theorems may be useful.

Theorem of Perpendicular axes: If $I_x \text{ and } I_y$ be the moments of inertia of a plane lamina of mass M about two axes $OX \text{ and } OY$ at right angles to each other in its plane, then the moment of inertia I_z of the lamina about the axis OZ perpendicular to the plane of the lamina is given by $I_z = I_x + I_y$.

Theorem of Parallel axes:

If I be the moment of inertia of a body of mass M about any axis CD and I_G, its moment of inertia about a parallel axis AB passing through the centre of gravity of the body and a, the distance between two axes, then $I = I_G + Ma^2$



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Solved Problems

1 Find the moment of inertia of an area bounded by the lines $\frac{x}{a}+\frac{y}{b}=1,\ x=0,\ y=0$ about the $z-$ axis.

Since the density is not given, assume that $\rho=k,$ a constant.

The mass of the area is given by

$$M=\iint\limits_{R} \rho\ dxdy$$

$$=k\int_0^b \int_0^{\frac{a}{b}(b-y)} dx\,dy$$

$$=k\int_0^b \frac{a}{b}(b-y)\ dy$$

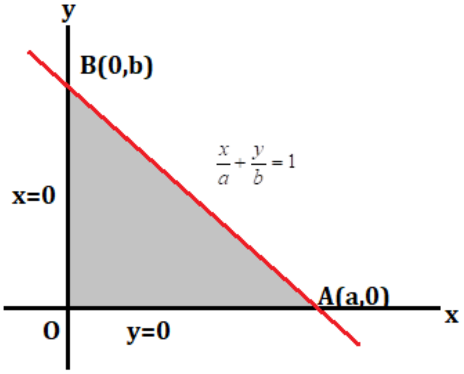
$$=k\frac{a}{b}\left(by-\frac{y^2}{2}\right)_0^b$$

$$=k\frac{a}{b}\left(b^2-\frac{b^2}{2}\right)$$

$$=k\frac{a}{b}.\frac{b^2}{2}$$

$$=\frac{kab}{2}$$

$$Hence\ k=\frac{2M}{ab}$$



From the given curve

$$\frac{x}{a}=1-\frac{y}{b},\ \ i.e.\ \frac{x}{a}=\frac{b-y}{b}$$

$$x=a\frac{b-y}{b},\ \ i.e.\ \ x=\frac{a}{b}(b-y)$$

For the area $OABO$,

y varies from $0\ to\ b$ & x varies from $0\ to\ \frac{a}{b}(b-y)$

Moment of Inertia about $x-$axis is given by

$$I_x=\iint\limits_{R} \rho y^2\ dxdy$$

$$=k\int_0^b \int_0^{\frac{a}{b}(b-y)} y^2\ dx\,dy$$

$$=k\int_0^b y^2\ \frac{a}{b}(b-y)\ dy$$

$$=k\frac{a}{b}\int_0^b \left(by^2-y^3\right)\ dy$$

$$=k\frac{a}{b}\left(b\frac{y^3}{3}-\frac{y^4}{4}\right)_0^b$$

Moment of Inertia about $y-$axis is given by

$$I_y=\iint\limits_{R} \rho x^2\ dxdy$$

$$=k\int_0^b \int_0^{\frac{a}{b}(b-y)} x^2\ dx\,dy$$

$$=k\int_0^b \left[\frac{x^3}{3}\right]_0^{\frac{a}{b}(b-y)} dy$$

$$=\frac{k}{3}\int_0^b \frac{a^3}{b^3}(b-y)^3\ dy$$

$$= k\frac{a}{b}\left[\frac{b^4}{3}-\frac{b^4}{4}\right]$$

$$= k\frac{a}{b}\frac{b^4}{12}$$

$$= \frac{kab^3}{12}$$

$$= \frac{2M}{ab}\cdot\frac{ab^3}{12} \quad (substitute\ k\ value)$$

$$= \frac{Mb^2}{6}$$

$$= \frac{k}{3}\frac{a^3}{b^3}\left[\frac{(b-y)^4}{-4}\right]_0^b$$

$$= -\frac{k}{3}\cdot\frac{a^3}{b^3}\cdot\frac{1}{4}\left[0-b^4\right]$$

$$= \frac{k}{3}\cdot\frac{a^3}{b^3}\cdot\frac{1}{4}b^4$$

$$= \frac{ka^3b}{12}$$

$$= \frac{2M}{ab}\cdot\frac{a^3b}{12} \quad (substitute\ k\ value)$$

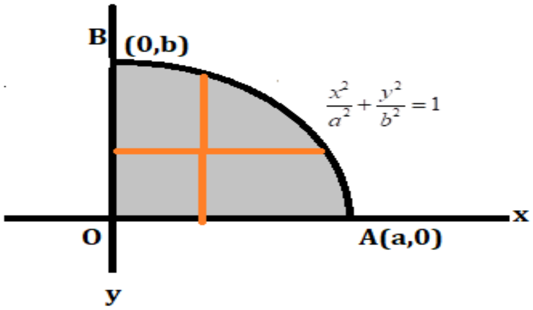
$$= \frac{Ma^2}{6}$$

By perpendicular axis theorem, the moment of inertia about $z-$axis is given by

$$I_z = I_x + I_y = \frac{Mb^2}{6}+\frac{Ma^2}{6}=\frac{M}{6}\left(a^2+b^2\right)$$

2 Find the moment of inertia of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of mass M about the $z-$ axis, if the density at a point is proportional to xy.

Given that density $\rho = kxy$. The mass of the quadrant is given by



From the given curve

$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}, \quad i.e.\ \frac{x^2}{a^2}=\frac{b^2-y^2}{b^2}$$

$$x^2 = a^2\frac{b^2-y^2}{b^2}, \quad i.e.\ x=\frac{a}{b}\sqrt{b^2-y^2}$$

For the area $OABO$,
y varies from $0\ to\ b$ & x varies from $0\ to\ \frac{a}{b}\sqrt{b^2-y^2}$

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$M = \iint_R \rho \, dxdy$ $= k\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy \, dx\,dy$ $= k\int_0^b y\left[\frac{x^2}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$ $= \frac{k}{2}\int_0^b y\,\frac{a^2}{b^2}\left(b^2-y^2\right)dy$ $= \frac{k}{2}\frac{a^2}{b^2}\int_0^b \left(b^2y-y^3\right)dy$ $= \frac{k}{2}\cdot\frac{a^2}{b^2}\left[b^2\frac{y^2}{2}-\frac{y^4}{4}\right]_0^b$ $= \frac{k}{2}\cdot\frac{a^2}{b^2}\left[\frac{b^4}{2}-\frac{b^4}{4}\right]$ $= \frac{ka^2b^2}{8}$ $\therefore k = \frac{8M}{a^2b^2}$	
Moment of Inertia about $x-$axis is given by	Moment of Inertia about $y-$axis is given by

$$I_x = \iint_R \rho y^2 \; dxdy$$

$$= k\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy^3 \; dx\,dy$$

$$= k \int_0^b y^3 \left[\frac{x^2}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \frac{k}{2}\int_0^b y^3 \; \frac{a^2}{b^2}\left(b^2-y^2\right) dy$$

$$= \frac{k}{2}\frac{a^2}{b^2}\int_0^b \left(b^2y^3-y^5\right) dy$$

$$= \frac{k}{2}.\frac{a^2}{b^2}\left[b^2\frac{y^4}{4}-\frac{y^6}{6}\right]_0^b$$

$$= \frac{k}{2}.\frac{a^2}{b^2}\left[\frac{b^6}{4}-\frac{b^6}{6}\right]$$

$$= \frac{k}{2}.\frac{a^2}{b^2}.\frac{b^6}{12}$$

$$= \frac{ka^2b^4}{24}$$

$$= \frac{8M}{a^2b^2}.\frac{a^2b^4}{24} \quad (substitute \; k \; value)$$

$$= \frac{Mb^2}{3}$$

$$I_y = \iint_R \rho x^2 \; dxdy$$

$$= k\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} y\,x^3 \; dx\,dy$$

$$= k \int_0^b y \left[\frac{x^4}{4}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \frac{k}{4}\int_0^b y \; \frac{a^4}{b^4}\left(b^2-y^2\right)^2 \; dy$$

$$= \frac{k}{4}\frac{a^4}{b^4}\int_0^b \frac{1}{2}\left(b^2-y^2\right)^2 \; d\left(y^2\right)$$

$$= \frac{k}{4}.\frac{a^4}{b^4}.\frac{1}{2}\left[\frac{\left(b^2-y^2\right)^3}{-3}\right]_0^b$$

$$= \frac{k}{4}.\frac{a^4}{b^4}.\frac{1}{-3\times 2}\left[0-b^6\right]$$

$$= \frac{k}{4}.\frac{a^4}{b^4}.\frac{1}{6}\,b^6$$

$$= \frac{ka^4b^2}{24}$$

$$= \frac{8M}{a^2b^2}.\frac{a^4b^2}{24} \quad (substitute \; k \; value)$$

$$= \frac{Ma^2}{3}$$

By perpendicular axis theorem, the moment of inertia about $z-$axis is given by

$$I_z = I_x + I_y = \frac{Ma^2}{3}+\frac{Mb^2}{3} = \frac{M}{3}\left(a^2+b^2\right)$$

3 Find the moment of inertia of a solid right circular cylinder about its axis and about a diameter of the base.

First let us find the mass of the cylinder.

$$Mass = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{0}^{h} \rho \ dz \ dy \ dx$$

$$= \rho \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} h \ dy \ dx$$

$$= \rho . h \int_{-a}^{a} 2\sqrt{a^2-x^2} \ dx$$

$$= 2\rho . h \ 2 \int_{0}^{a} \sqrt{a^2-x^2} \ dx, \quad \sin ce \ the \ \text{int} egrand \ is \ even$$

$$= 4\rho h \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a$$

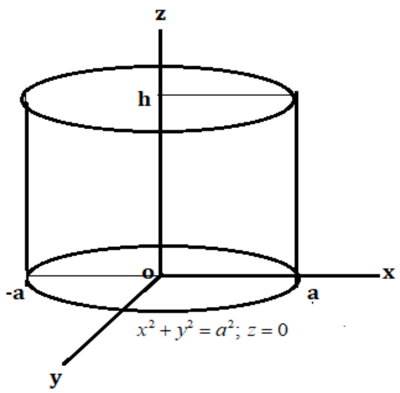
$$= 4\rho h \left[\frac{a^2}{2}\sin^{-1} 1 \right], \quad \sin^{-1} 0 = 0$$

$$= 4\rho h \left[\frac{a^2}{2}\sin^{-1} 1 \right], \quad \sin^{-1} 1 = \frac{\pi}{2}$$

$$= 4\rho h \frac{a^2}{2} \frac{\pi}{2}$$

$$= \rho h \pi a^2$$

$$Hence, \ \rho = \frac{M}{h\pi a^2}$$



Let the cylinder is defined by
$x^2 + y^2 = a^2; z = 0; z = h$

Here x varies from $-a \ to \ a$ and y varies from $-\sqrt{a^2-x^2} \ to \ \sqrt{a^2-x^2}$ and height varies from $z = 0 \ to \ z = h$.

Let V be the volume, M be the mass and ρ be the density.

To find moment of inertia of the cylinder about its axis($z-$axis)

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$$I_z = \iiint\limits_V \rho\left(x^2+y^2\right)dxdydz$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{h} \rho\left(x^2+y^2\right)dz\ dy\ dx$$

$$= \rho\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} h\left(x^2+y^2\right)dy\ dx$$

$$= 2\rho h\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \left(x^2+y^2\right)dy\ dx,\ Even\ function\ property$$

$$= 2\rho h\int_{-a}^{a}\left(x^2y+\frac{y^3}{3}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= 2\rho h\int_{-a}^{a}\left(x^2\sqrt{a^2-x^2}+\frac{\left(a^2-x^2\right)^{\frac{3}{2}}}{3}\right)dx$$

$$= 2\rho h\int_{-a}^{a} x^2\sqrt{a^2-x^2}\ dx+\frac{2}{3}\rho h\int_{-a}^{a}\left(a^2-x^2\right)^{\frac{3}{2}}dx$$

$$= 4\rho h\int_{0}^{a} x^2\sqrt{a^2-x^2}\ dx+\frac{4}{3}\rho h\int_{0}^{a}\left(a^2-x^2\right)^{\frac{3}{2}}dx$$

$$\left[put\ x=a\sin\theta,\ x=0\Rightarrow\theta=0,\ x=0\Rightarrow\theta=\pi/2,\ dx=a\cos\theta\right]$$

$$= 4\rho h\int_{0}^{\frac{\pi}{2}} a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}\ a\cos\theta\ d\theta+\frac{4}{3}\rho h\int_{0}^{\frac{\pi}{2}}\left(a^2-a^2\sin^2\theta\right)^{\frac{3}{2}}a\cos\theta\ d\theta$$

$$= 4\rho h\int_{0}^{\frac{\pi}{2}} a^4\sin^2\theta\sqrt{1-\sin^2\theta}\ \cos\theta\ d\theta+\frac{4}{3}\rho h\int_{0}^{\frac{\pi}{2}} a^4\left(1-\sin^2\theta\right)^{\frac{3}{2}}\cos\theta\ d\theta$$

$$= 4\rho ha^4\int_{0}^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\ d\theta+\frac{4}{3}\rho ha^4\int_{0}^{\frac{\pi}{2}}\cos^4\theta\ d\theta$$

$$= 4\rho ha^4\left[\frac{1}{4.2}\frac{\pi}{2}\right]+\frac{4}{3}\rho ha^4\left[\frac{3.1}{4.2}\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4}\rho ha^4+\frac{\pi}{4}\rho ha^4$$

$$= \frac{\pi}{2}\rho ha^4$$

$$= \frac{\pi}{2}ha^4\times\frac{M}{h\pi a^2},\qquad \sin ce\ \rho=\frac{M}{h\pi a^2}$$

$$= \frac{M}{2}a^2$$

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To find moment of inertia of the cylinder about its diameter of the base(say $y-$axis)

$$I_y = \iiint_V \rho\left(x^2+z^2\right)dxdydz$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{h} \rho\left(x^2+z^2\right)\,dz\;dy\;dx$$

$$= \rho\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(x^2 z+\frac{z^3}{3}\right)_0^h\;dy\;dx$$

$$= \rho\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(x^2 h+\frac{h^3}{3}\right)dy\;dx$$

$$= \rho\int_{-a}^{a} \left(x^2 h+\frac{h^3}{3}\right)2\sqrt{a^2-x^2}dx$$

$$= 2\rho h\int_{-a}^{a} x^2\sqrt{a^2-x^2}dx+\frac{2}{3}\rho h^3\int_{-a}^{a}\sqrt{a^2-x^2}dx$$

$$= 4\rho h\int_{0}^{a} x^2\sqrt{a^2-x^2}dx+\frac{4}{3}\rho h^3\int_{0}^{a}\sqrt{a^2-x^2}dx$$

$$\left[put\;x=a\sin\theta,\;x=0\Rightarrow\theta=0,\;x=0\Rightarrow\theta=\pi/2,\;dx=a\cos\theta\right]$$

$$= 4\rho h\int_{0}^{\frac{\pi}{2}} a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\;d\theta+\frac{4}{3}\rho h^3\int_{0}^{\frac{\pi}{2}}\left(a^2-a^2\sin^2\theta\right)^{\frac{1}{2}}a\cos\theta\;d\theta$$

$$= 4\rho h\int_{0}^{\frac{\pi}{2}} a^4\sin^2\theta\sqrt{1-\sin^2\theta}\;\cos\theta\;d\theta+\frac{4}{3}\rho h^3\int_{0}^{\frac{\pi}{2}} a^2\left(1-\sin^2\theta\right)^{\frac{1}{2}}\cos\theta\;d\theta$$

$$= 4\rho h a^4\int_{0}^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\;d\theta+\frac{4}{3}\rho h^3 a^2\int_{0}^{\frac{\pi}{2}}\cos^2\theta\;d\theta$$

$$= 4\rho h a^4\left[\frac{1}{4.2}\frac{\pi}{2}\right]+\frac{4}{3}\rho h^3 a^2\left[\frac{1}{2}\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4}\rho h a^4+\frac{4}{3}\rho h^3 a^2\frac{\pi}{4}$$

$$= \rho\;h a^2\frac{\pi}{4}\left[a^2+\frac{4}{3}h^2\right]$$

$$= \frac{M}{h\pi a^2}h a^2\frac{\pi}{4}\left[\frac{3a^2+4h^2}{3}\right],\qquad \sin ce\;\;\rho=\frac{M}{h\pi a^2}$$

$$= \frac{M}{12}\left[3a^2+4h^2\right]$$

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4 Find the moment of inertia of the homogeneous solid bounded by the cylinder $x^2+y^2=a^2;\ z=0;\ z=h$ about the $x-$axis.

We know that the mass and the density of the given cylinder is $M=\rho h\pi a^2\ \ \rho=\dfrac{M}{h\pi a^2}$ respectively. Refer the previous example. Now let us find the moment of inertia about $x-$axis.

$$I_x=\iiint\limits_V \rho\left(y^2+z^2\right)dxdydz$$

$$=\int\limits_{-a}^{a}\int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int\limits_{0}^{h}\rho\left(y^2+z^2\right)dz\ dy\ dx$$

$$=\rho\int\limits_{-a}^{a}\int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(y^2z+\frac{z^3}{3}\right)_0^h dy\ dx$$

$$=\rho\int\limits_{-a}^{a}\int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(y^2h+\frac{h^3}{3}\right)dy\ dx$$

$$=2\rho\int\limits_{-a}^{a}\int\limits_{0}^{\sqrt{a^2-x^2}}\left(y^2h+\frac{h^3}{3}\right)dy\ dx,\ \because \text{int}\,egrand\ is\ even\ in\ y$$

$$=2\rho\int\limits_{-a}^{a}\left[h\frac{y^3}{3}+\frac{h^3}{3}y\right]_0^{\sqrt{a^2-x^2}}dx$$

$$=2\rho\int\limits_{-a}^{a}\left[\frac{h}{3}\left(a^2-x^2\right)^{\frac{3}{2}}+\frac{h^3}{3}\sqrt{a^2-x^2}\right]dx$$

$$=2\rho\frac{h}{3}\int\limits_{-a}^{a}\left(a^2-x^2\right)^{\frac{3}{2}}dx+2\rho\frac{h^3}{3}\int\limits_{-a}^{a}\sqrt{a^2-x^2}\ dx$$

$$=4\rho\frac{h}{3}\int\limits_{0}^{a}\left(a^2-x^2\right)^{\frac{3}{2}}dx+4\rho\frac{h^3}{3}\int\limits_{0}^{a}\sqrt{a^2-x^2}\ dx$$

$$\left[put\ x=a\sin\theta,\ x=0\Rightarrow\theta=0,\ x=0\Rightarrow\theta=\pi/2,\ dx=a\cos\theta\right]$$

$$=\frac{4}{3}\rho h\int\limits_{0}^{\frac{\pi}{2}}\left(a^2-a^2\sin^2\theta\right)^{\frac{3}{2}}a\cos\theta\ d\theta+\frac{4}{3}\rho h^3\int\limits_{0}^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\theta}\ a\cos\theta\ d\theta$$

$$=\frac{4}{3}\rho h\int\limits_{0}^{\frac{\pi}{2}}a^4\left(1-\sin^2\theta\right)^{\frac{3}{2}}\cos\theta\ d\theta+\frac{4}{3}\rho h^3\int\limits_{0}^{\frac{\pi}{2}}a^2\left(1-\sin^2\theta\right)^{\frac{1}{2}}\cos\theta\ d\theta$$

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$$=\frac{4}{3}\rho ha^4\int_0^{\frac{\pi}{2}} \cos^4\theta\ d\theta+\frac{4}{3}\rho h^3a^2\int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

$$=\frac{4}{3}\rho ha^4\left[\frac{3.1}{4.2}\frac{\pi}{2}\right]+\frac{4}{3}\rho h^3a^2\left[\frac{1}{2}\frac{\pi}{2}\right]$$

$$=\frac{4}{3}.\frac{3}{4}.\frac{\pi}{4}\rho ha^4+\frac{4}{3}\rho h^3a^2.\frac{\pi}{4}$$

$$=\frac{4}{3}.\frac{\pi}{4}\rho ha^2\left[\frac{3a^2}{4}+h^2\right]$$

$$=\frac{M}{h\pi a^2}.\frac{4}{3}.\frac{\pi}{4}.ha^2\left[\frac{3a^2+4h^2}{4}\right], \qquad \sin ce\ \ \rho=\frac{M}{h\pi a^2}$$

$$=\frac{M}{12}\left[3a^2+4h^2\right]$$

Exercise

1 Find the mass of a solid in the form of the positive octant of the sphere $x^2+y^2+z^2=9$, if the density at any point is $2xyz$.

2 A lamina is bounded by the curves $y=x^2-3x\ andy=2x.$ If the density at a point is given by kxy, find by double integration, the mass of the lamina.

3 Find the mass of a lamina in the form of the cardioid $r=a(1+\cos\theta)$ whose density at any point varies as the square of its distance from the initial line.

4 Find by double integration, the centre of gravity of the area of the cardioid $r=a(1+\cos\theta)$

5 Find the centroid of the area enclosed by the parabola $y^2=4ax$, the x-axis and its latus rectum.

6 The density at any point of a lamina is $k(x+y)$. The lamina is bounded by the lines $x=0, x=a, y=0, y=b.$ Find the position of centre of gravity.

7 Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 5 mtrs and 4 mtrs respectively.

8 Find the moment of inertia of the area bounded by the curve $r^2=a^2\cos 2\theta$ about its axis

9 Find the moment of inertia of a circular plate about a tangent.

10. Find the moment of inertia of the area $y=\sin x$ from $x=0\ to\ x=2\pi$ about x-axis.

<u>Integration Formulas:</u>

Trigonometric Forms

$$\int \sin ax\,dx = -\frac{\cos ax}{a} + c$$

$$\int \cos ax\,dx = \frac{\sin ax}{a} + c$$

$$\int \tan x\,dx = \log\sec x + c$$

$$\int \cot x\,dx = \log\sin x + c$$

$$\int \sec x\,dx = \log(\sec x + \tan x) + c$$

$$\int \cos ecx\,dx = -\log(co\sec x + \cot x) + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int co\sec^2 x\,dx = -\cot x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int co\sec x \cot x\,dx = -co\sec x + c$$

$$\int co\sec x \cot x\,dx = -co\sec x + c$$

Heperbolic Forms

$$\int \sinh x\,dx = \cosh x + c$$

$$\int \cosh x\,dx = \sinh x + c$$

$$\int \tanh x\,dx = \log\cosh x + c$$

Forms Involving $a^2 - x^2$

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\frac{x+a}{x-a} + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

Inverse Trigonometric Forms

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1 - x^2} + c$$

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1 - x^2} + c$$

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\log\left(1 + x^2\right) + c$$

Exponential and Logarithmic Forms

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$$

$$\int a^x dx = a^x \log_a e + c$$

$$\int \frac{1}{x}\,dx = \log x + c$$

$$\int \log x\,dx = x\log x - x + c$$

Forms Involving $a^2 + x^2$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sinh^{-1}\frac{x}{a} = \log\left[x + \sqrt{a^2 + x^2}\right]$$

$$\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$$

Forms Involving $x^2 - a^2$

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\frac{x-a}{x+a} + c$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \cosh^{-1}\frac{x}{a} = \log\left[x + \sqrt{x^2 - a^2}\right]$$

$$\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c$$

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Other Formulas	Special Formulas
$\int k\,dx = kx + c$ $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ $\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$ $\int \frac{x}{\sqrt{x^2+a^2}}\,dx = \sqrt{x^2+a^2} + c$	$\int_0^\infty x e^{-x}\,dx = 1!$ $\int_0^\infty e^{-x} x^{n-1}\,dx = \lceil n$ $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\,d\theta = \frac{(m-1)(m-3)......1(n-1)(n-3).......1}{(m+n)(m+n-2)(m+n-4)........2}\frac{\pi}{2}, \mathit{if}\ \ m,n\ \mathit{is\,even}$ $\int_{-a}^{a} f(x)\,dx = 0\ \mathit{if}\ \ f(x)\ \mathit{is}\ \mathit{odd}$ $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx\,\mathit{if}\ \ f(x)\ \mathit{is}\ \mathit{even}$ $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx\,\mathit{if}\ \ f(2a-x) = f(x)$

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